

# ABELIAN PROPERTIES OF ANICK SPACES

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The Anick space,  $T_{2n-1}(p^r)$  is a  $p$ -local  $H$ -space with  $Z/p$  homology a symmetric algebra:<sup>1</sup>

$$H_*(T; Z/p) \cong Z/p[v, 2n] \otimes \Lambda(u, 2n-1)$$

where  $\beta^r(v) = u$ . These spaces were first constructed in [Ani93] for  $p > 3$  and shown to admit an  $H$ -space structure in [AG95]. A much simpler construction together with the  $H$ -space structure was obtained in [GT10] for all  $p > 2$ . The Anick spaces lie in an  $H$ -fibration sequence

$$\Omega^2 S^{2n+1} \xrightarrow{\pi_n} S^{2n-1} \longrightarrow T \longrightarrow \Omega S^{2n+1}$$

where the map  $\pi_n$  is such that the composition with the double suspension

$$\Omega^2 S^{2n+1} \xrightarrow{\pi_n} S^{2n-1} \longrightarrow \Omega^2 S^{2n+1}$$

is the  $p^r$ th power map. Such maps were first discovered by Cohen, Moore and Neisendorfer [CMN79b] where the authors used this factorization to determine that the exponent for the  $p$  torsion in  $\pi_*(S^{2n+1})$  is  $p^r$  when  $p > 2$ .

By an Abelian  $H$ -space we mean a space with an  $H$ -space structure which is homotopy associative and homotopy commutative. The purpose of this work is to prove:

**Main Theorem.** *If  $p > 3$ , the Anick spaces admit an Abelian  $H$ -space structure.*<sup>2</sup>

This may not seem surprising, but obtaining a proof involved several new techniques, and this work is as much about these techniques as about this result. Previous attempts to prove this ([The01]) were unsuccessful despite published announcements.

This result has implications, however, that are not immediately transparent, and in order to explain this we need to discuss some background.

## EHP SPECTRA

The sphere spectrum is a homotopy commutative ring spectrum where the ring structure can be defined by the composition of maps between spheres. Unstably, commutators are Whitehead products of the respective Hopf invariants, by the Barratt–Toda formula [Bar61]. The structure and the workings of the EHP sequences are determined by the resulting composition

<sup>1</sup>We will abbreviate  $T_{2n-1}(p^r)$  as  $T$  if there will be no confusion.

<sup>2</sup>If  $p = 3$ , no such structure can exist unless  $n = p^i$  for some values of  $i$ . See Appendix A.

theory [Tod56]. It is natural to ask whether other homotopy commutative ring spectra can have a system of unstable approximations with Hopf invariants fitting together in this way. After some calculations, the cyclic spectra of Smith–Toda type appear to be suitable candidates ([Gra93a], [Gra93b], [Gra98]).

The first example to study is  $S^0/p^r = S^0 \cup_{p^r} e^1$ , and we were led to conjecture a sequence of  $(n-1)$ -connected approximations,  $T_n(p^r)$  and secondary EHP sequences defined by fibrations:

$$\begin{aligned} T_{2n-1}(p^r) &\longrightarrow \Omega T_{2n}(p^r) \longrightarrow \Omega T_{2np-1}(p) \\ T_{2n}(p^r) &\longrightarrow \Omega T_{2n+1}(p^r) \longrightarrow \Omega T_{2(n+1)p-1}(p). \end{aligned}$$

These can be thought of as differential equations, and if we start with  $T_0 = Z/p^r$  (with the discrete topology), we can inductively see that the  $Z/p$  homology of  $T_n$  is a symmetric algebra on classes  $u, v$  with  $|u| = n$  and  $\beta^{(n)}(v) = u$ . One solution, then, is to set  $T_{2n} = S^{2n+1}\{p^r\}$ , the fiber of the degree  $p^r$  map on  $S^{2n+1}$ .

In [GT10], Hopf invariants were defined and EHP fibrations were obtained<sup>3</sup>

$$\begin{aligned} T_{2n-1}(p^r) &\longrightarrow \Omega T_{2n}(p^r) \xrightarrow{H} BW_n \\ T_{2n}(p^r) &\longrightarrow \Omega T_{2n+1}(p^r) \xrightarrow{H'} BW_{n+1} \end{aligned}$$

where  $BW_n$  is the classifying space for the fiber of the double suspension ([Gra88]):

$$W_n \longrightarrow S^{2n-1} \longrightarrow \Omega^2 S^{2n+1} \xrightarrow{\nu} BW_n.$$

The rationale behind seeking such EHP sequences is that they were suggested by a corresponding unstable composition theory. Write  $P^m = S^{m-1} \cup_{p^r} e^m$  and suppose that we are given a stable map:  $P^m \longrightarrow P^0$ . Then, for some  $n$ , there is an unstable representative:

$$P^{m+n} \longrightarrow T_{n-1}(p^r).$$

An unstable composition would then be a pairing:

$$[P^{m+n}, T_{n-1}(p^r)] \times [P^n, T_k(p^r)] \rightarrow [P^{m+n}, T_k(p^r)].$$

In order to effect such a pairing, we would need to have a natural extension:

$$\begin{array}{ccc} P^n & \xrightarrow{\beta} & T_k(p^r) \\ \downarrow & \nearrow \beta & \\ T_{n-1}(p^r) & & \end{array}$$

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<sup>3</sup>Note that we have replaced  $\Omega T_{2n-1}(p)$  with  $BW_n$ . We conjecture that these spaces are homotopy equivalent.

In [Gra93a, 3.4] it was proved that if  $Y$  is an Abelian  $H$ -space and  $\beta: P^{2n+1}(p^r) \rightarrow Y$ , there is a unique  $H$ -map  $\hat{\beta}: T_{2n}(p^r) \rightarrow Y$ , and in [Nei83] it was proved that if  $p > 3$ ,  $T_{2n}(p^r)$  has an Abelian  $H$ -space structure, unique up to an equivalence. The result of the theorem, then, is that such a map  $\hat{\beta}$  uniquely exists for each  $k$  when  $n$  is odd. The case of  $n$  even is as yet unsettled. In [AG95], it was proved that such a map  $\hat{\beta}$  exists but it may not be an  $H$ -map and it may not be unique, while in [Gra12] it was proved that if  $\hat{\beta}$  is an  $H$ -map it is unique.

### OUTLINE OF PROOF

We separate the case  $n = 1$  from the cases  $n > 1$ . In case  $n = 1$ ,  $T_{2n-1}(p^r)$  is actually a double loop space (See Appendix A), so we assume  $n > 1$ . This hypothesis will be needed in section 5.

The method is a refinement of that used in [GT10], which we review in section 2. In that work an extension theorem was developed ([GT10, 2.1]). Given a principal fibration

$$\Omega X \longrightarrow E \xrightarrow{\pi} B \cup_{\phi} CA$$

the extension theorem allowed any map  $\nu_0: \pi^{-1}(B) \rightarrow BW_n$  to be extended to  $E$  under suitable conditions on the attaching map  $\phi$ . This result was applied inductively choosing an arbitrary extension in each case. From this the authors constructed a fibration

$$(1.1) \quad \Omega G \xrightarrow{h} T \longrightarrow R \longrightarrow G$$

in which  $h$  has a right homotopy inverse. However the choices of the extensions have a direct impact on the resulting  $H$ -space structure on  $T$ . (See section 2 for more details.)

In this work we introduce a controlled extension theorem 8.1. We bring more information to bear and exercise a more limited choice for each extension. In particular, we will construct a collection of  $\text{mod } p^r$  homotopy classes and our extensions will be required to annihilate these classes. We will refer to these classes as the obstructions. The extensions we construct here are among the possibilities in [GT10], so all the results in that work apply here as well.

### CO- $H$ SPACES AND GENERALIZED WHITEHEAD PRODUCTS

The fibration (1.1) is constructed in [AG95] as well as [GT10]; in both works it is proved that  $T$  is a retract of  $\Omega G$  and  $G$  is a retract of  $\Sigma T$ . In particular,  $G$  has a co- $H$  space and is filtered by subspaces  $G_k$  where

$$G_k = G_{k-1} \cup CP^{2np^k}(p^{r+k})$$

with  $G_0 = P^{2n+1}$ .

In [Gra11], a new kind of Whitehead product was introduced. Given any two simply connected co- $H$  spaces  $G, H$ , a cofibration sequence

$$G \circ H \xrightarrow{W} G \vee H \longrightarrow G \times H$$

was constructed, where  $G \circ H$  is a new co- $H$  space which depends functorially on  $G$  and  $H$ . This generalizes the usual map  $SX \wedge Y \xrightarrow{W} SX \vee SY$  when  $G = SX$  and  $H = SY$ , which is used to define Whitehead product. However, we can write  $G^{[j]} = G \circ G^{[j-1]}$  for  $j > 1$  and define an iterated Whitehead product

$$ad^j: G^{[j]} \rightarrow G$$

as the composition

$$G^{[j]} = G \circ G^{[j-1]} \xrightarrow{W} G \vee G^{[j-1]} \xrightarrow{1 \vee ad^{j-1}} G$$

This is of particular interest when  $G = G_k$ . In [GT10] the authors defined a map  $\varphi_k: G_k \rightarrow S^{2n+1}\{p^r\}$  from which they constructed a principal fibration:

$$\Omega S^{2n+1}\{p^r\} \longrightarrow E_k \longrightarrow G_k.$$

Functorial maps  $G_k^{[j]} \xrightarrow{\widetilde{ad}^j} E_k$  are defined which cover  $ad^j: G_k^{[j]} \rightarrow G_k$ . These maps are the precursors to the obstructions and the task is to construct  $\nu_k: E_k \rightarrow BW_n$  such that  $\nu_k \widetilde{ad}^j \sim *$  for  $j \geq 2$ .

#### APPROXIMATION

The iterated products  $ad^j: G_k^{[j]} \rightarrow G_k$  are difficult to work with because of the complexity of  $G_k$ . However they can be sufficiently approximated by iterated Whitehead products defined on Moore's spaces. It is in this section that we require  $n \neq 1$ . In [Nei10a], the author constructed  $H$ -space based Whitehead products

$$\pi_{m+1}(B; Z/p^r) \otimes \pi_{n+1}(B; Z/p^r) \rightarrow \pi_{m+n+1}(E; Z/p^r)$$

in the case of a principal fibration classified by a map  $\varphi: B \rightarrow X$  where  $X$  is a homotopy commutative  $H$ -space. The maps  $\widetilde{ad}^j$  are then replaced by these  $H$ -space based Whitehead products.

#### CONGRUENCE HOMOTOPY THEORY

Define two maps  $f, g: X \rightarrow Y$  to be congruent if  $\Sigma f \sim \Sigma g$ . Since  $BW_n$  is an  $H$ -space, we need only consider the congruence classes of the obstructions. They are all  $H$ -space based Whitehead products, but except for a few, they are all congruent to relative Whitehead products.

Given a principal fibration

$$\Omega X \longrightarrow E \longrightarrow B$$

the relative Whitehead product is a pairing

$$\pi_{n+1}(B; Z/p^r) \otimes \pi_m(E; Z/p^r) \rightarrow \pi_{m+n}(E; Z/p^r),$$

but if we consider only the congruence classes of mod  $p^r$  homotopy of  $E$  (written  $e\pi_m(E; Z/p^r)$ ) we actually get an action (7.9)

$$A_*(B) \otimes e\pi_*(E; Z/p^r) \rightarrow e\pi_*(E; Z/p^r),$$

where  $A_*(B)$  is the symmetric algebra generated by  $M_n = \pi_{n+1}(B; Z/p^r)$  in which the Bockstein is a derivation. Furthermore, this action commutes with the action of self maps of the Moore spaces (7.12)

$$\{e\pi_k(P^m; Z/p^r)\}.$$

### REDUCTION

We further reduce the number of obstructions by constructing a variant  $J_k$  of  $E_k$  defined over a variant  $D_k$  of  $G_k$

$$\begin{array}{ccc} E_k & \xrightarrow{\tau_k} & J_k \\ \downarrow & & \downarrow \\ G_k & \longrightarrow & D_k. \end{array}$$

$J_k$  is also the total space of a principal fibration.  $D_k$  and an associated space  $F_k$  were originally constructed by Anick [Ani93] and we reconstruct them here. Half of the obstructions are in the kernel of  $\tau_k$ , so we are reduced to constructing  $\gamma_k: J_k \rightarrow BW_n$  and then  $\nu_k = \gamma_k \tau_k$ . There is another principal fibration

$$\Omega^2 S^{2n+1} \longrightarrow J_k \longrightarrow F_k$$

and there are exactly two remaining obstructions corresponding to each cell of  $F_k$ . We apply the controlled extension theorem here to construct  $\gamma_k$  and hence  $\nu_k$ .

Unfortunately, although there is a natural map from  $J_{k-1} \xrightarrow{\iota} J_k$ , the maps  $\gamma_k \iota$  and  $\gamma_{k-1}$  only agree up to dimension  $2np^k$ . This is a serious problem as higher dimensional obstructions in  $J_{k-1}$  may not be in the kernel of  $\gamma_k$ . The solution to this has required another application of congruence homotopy theory. In the critical dimension, there is a formula relating these elements, and the  $A_*(D_k)$  module action allows one to propagate this formula to higher dimensions. This is a delicate point, and much care was needed here.

Several interesting technical results are included, and, in particular, it is shown that  $T$  has  $H$ -space exponent  $p^r$ .

I would like to thank Joe Neisendorfer, who has been a helpful correspondent throughout this work.

## NOTATION

I would like to set up some notational conventions. First, we will assume that all spaces are localized at a prime  $p > 2$  and often assume  $p > 3$ . We will write

$$P^m(p^s) = S^{m-1} \cup_{p^s} e^m$$

for the Moore space. We will always have  $s \geq r$  where  $r$  is fixed and we abbreviate  $P^m(p^r)$  as  $P^m$ . We write  $\iota_{m-1}$  and  $\pi_m$  for the standard maps

$$S^{m-1} \xrightarrow{\iota_{m-1}} P^m(p^s) \xrightarrow{\pi_m} S^m$$

and

$$\begin{aligned} \beta &: P^m(p^s) \rightarrow P^{m+1}(p^s) \\ \sigma &: P^m(p^s) \rightarrow P^m(p^{s-1}) \\ \rho &: P^m(p^s) \rightarrow P^m(p^{s+1}) : \end{aligned}$$

for the Bockstein and coefficient change maps; we write  $\rho^t$  and  $\sigma^t$  for iterations of these maps. We suppress the index  $m$  and the exponent  $s$  from the notation; in particular, we have formulas:

$$\begin{aligned} \beta &= \sigma\beta\rho \\ p &= \sigma\rho = \rho\sigma. \end{aligned}$$

We frequently will use  $\delta_t = \beta\rho^t$ .

Throughout the literature there have been various indexing conventions for the Anick spaces; we reserve the notation  $T^m$  for the  $m$  skeleton of  $T$ , and use a subscript, when necessary to indicate which Anick space is under consideration. We follow Mahowald's suggestion that the subscript should indicate the dimension of the bottom cell. If we need to indicate the prime power we write  $T(p^r)$  or  $T_{2n-1}(p^r)$  as needed.

## 2. REVIEW AND MODIFICATION OF [GT10]

The following diagram of fibrations and spaces can be found in [CMN79b]:

$$\begin{array}{ccccccc} \Omega^2 S^{2n+1} & \longrightarrow & \Omega S^{2n+1} \{p^r\} & \longrightarrow & \Omega S^{2n+1} & \xrightarrow{p^r} & \Omega S^{2n+1} \\ \parallel & & \downarrow & & \downarrow & & \parallel \\ \Omega^2 S^{2n+1} & \longrightarrow & E & \longrightarrow & F & \longrightarrow & \Omega S^{2n+1} \\ & & \downarrow & & \downarrow & & \\ & & P^{2n+1} & \xlongequal{\quad} & P^{2n+1} & & \\ & & \downarrow \varphi_0 & & \downarrow & & \\ & & S^{2n+1} \{p^r\} & \longrightarrow & S^{2n+1} & & \end{array}$$

The  $2n$  skeleton of  $F$  is  $S^{2n}$  and the restriction of the middle horizontal fibration by the inclusion  $S^{2n} \longrightarrow F$  yields a fibration

$$\Omega^2 S^{2n+1} \longrightarrow E(1) \longrightarrow S^{2n} \xrightarrow{\iota} \Omega^2 S^{2n+1}.$$

This fibration has been studied in [Gra88], where it is shown that the  $E(1) \simeq S^{4n-1} \times BW_n$ , and that  $BW_n$  is the classifying space of the fiber of the double suspension:

$$W_n \longrightarrow S^{2n-1} \xrightarrow{\iota} \Omega^2 S^{2n+1} \xrightarrow{\nu} BW_n$$

where  $\nu$  is the composition

$$\Omega^2 S^{2n+1} \longrightarrow E(1) \simeq S^{4n-1} \times BW_n \xrightarrow{\pi_2} BW_n.$$

This defines  $i: BW_n \rightarrow E$ , and in [GT10], the authors construct a retraction  $\nu^E: E \rightarrow BW_n$ .

Given such a retraction  $\nu^E$ , let  $H$  be the composition:

$$\Omega S^{2n+1} \{p^r\} \longrightarrow E \xrightarrow{\nu^E} BW_n.$$

One then has two diagrams of fibrations:

$$\begin{array}{ccccc} T & \xrightarrow{E} & \Omega S^{2n+1} \{p^r\} & \xrightarrow{H} & BW_n \\ \downarrow & & \downarrow & & \parallel \\ R_0 & \longrightarrow & E & \xrightarrow{\nu^E} & BW_n \\ \downarrow & & \downarrow & & \\ P^{2n+1} & \xlongequal{\quad} & P^{2n+1} & & \end{array}$$

(2.2)

$$\begin{array}{ccccc} \Omega^2 S^{2n+1} & \xlongequal{\quad} & \Omega^2 S^{2n+1} & & \\ \pi_n \downarrow & & \downarrow & & \\ S^{2n-1} & \longrightarrow & \Omega^2 S^{2n+1} & \xrightarrow{\nu} & BW_n \\ \downarrow & & \downarrow & & \parallel \\ T & \longrightarrow & \Omega S^{2n+1} \{p^r\} & \xrightarrow{H} & BW_n \\ \downarrow & & \downarrow & & \\ \Omega S^{2n+1} & \xlongequal{\quad} & \Omega S^{2n+1} & & \end{array}$$

The lower diagram presents the Anick space  $T$  together with its associated fibrations. The upper diagram is a first approximation to the fibration (1.1). The inclusion  $T \rightarrow R_0$  is only null homotopic on the  $2np - 2$  skeleton of  $T$ .

The remainder of [GT10] consists of adding cells to  $P^{2n+1}$  so that the corresponding map is null homotopic on successively larger skeleta of  $T$ . In general, a space  $G_k$  is obtained from  $G_{k-1}$  by attaching a Moore space:

$$G_k = G_{k-1} \cup CP^{2np^k}(p^{r+k})$$

with  $G_0 = P^{2n+1}$ , and the map  $\varphi_0$  is extended to a map

$$\varphi_k: G_k \rightarrow S^{2n+1}\{p^r\}.$$

This leads to a generalization of (2.2). Consider the inclusion:

$$BW_n \longrightarrow E = E_0 \longrightarrow E_k.$$

Given a retraction  $\nu_k: E_k \rightarrow BW_n$ , we obtain:

$$\begin{array}{ccccc} T & \longrightarrow & \Omega S^{2n+1}\{p^r\} & \xrightarrow{H} & BW_n \\ \downarrow & & \downarrow & & \parallel \\ R_k & \longrightarrow & E_k & \xrightarrow{\nu_k} & BW_n \\ \downarrow & & \downarrow & & \\ G_k & \xlongequal{\quad} & G_k & & \\ & & \downarrow \varphi_k & & \\ & & S^{2n+1}\{p^r\} & & \end{array}$$

We will need to consider various possible retractions  $\nu_k$ . For any choice, the following properties were proved in [GT10]:

- (A) The composition:  $T^{2np^k-2} \rightarrow T \rightarrow R_{k-1}$  is null homotopic ([GT10, 4.3(b)]);
- (B)  $\Sigma(T \wedge T)$  is a wedge of Moore spaces ([GT10, 4.5]);
- (C)  $G$  is an atomic retract of  $\Sigma T$  ([GT10, 4.7]);
- (D)  $R$  is a wedge of Moore spaces, where  $R = \varinjlim R_k$  ([GT10, 4.8]);
- (E) For any choice of  $H$ -space structure, the map  $T \rightarrow \Omega S^{2n+1}\{p^r\}$  is an  $H$ -map ([GT10, 4.6]).

This leads to a fibration:

$$\Omega G \xrightarrow{*} T \longrightarrow R \longrightarrow G.$$

Different choices for  $\nu_k$  determine different maps  $R \rightarrow G$ . These choices have a direct effect on whether the inherited  $H$ -space structure is Abelian.

We now turn to the question of how to make beneficial choices. For this purpose, recall that the homotopy fiber of the inclusion

$$X \vee Y \rightarrow X \times Y$$

is homotopy equivalent to  $\Omega X * \Omega Y$  ([Gan70],[Gra71]), yielding a fibration sequence:

$$\Omega X * \Omega Y \xrightarrow{\omega} X \vee Y \longrightarrow X \times Y.$$



The composition

$$\Omega B * \Omega B \xrightarrow{\omega} B \vee B \xrightarrow{\nabla} B,$$

where  $\nabla$  is the folding map, is sometimes called the universal Whitehead product map, since if we are given maps  $\alpha: \Sigma X \rightarrow B$ ,  $\beta: \Sigma Y \rightarrow B$ , the composition

$$\{\alpha, \beta\} \cdot \Sigma X \wedge Y \xrightarrow{W} \Sigma X \vee \Sigma Y \xrightarrow{\alpha \vee \beta} B$$

factors through  $\nabla\omega$ .

Given any fibration

$$\Omega B \xrightarrow{\partial} F \xrightarrow{i} E \xrightarrow{\pi} B$$

in which  $i$  is null homotopic, Theriault discovered a necessary and sufficient condition for the  $H$ -space structure defined by any right inverse to  $\partial$  be homotopy Abelian.

**Theorem 2.3** (Theriault criterion). [The01, 4.2] *The  $H$ -space structure defined on  $F$  by any right inverse to  $\partial$  is Abelian iff the universal Whitehead product on  $B$  lifts to  $E$*

$$\begin{array}{ccccc} & & & & E \\ & & & \nearrow & \downarrow \pi \\ \Omega B * \Omega B & \xrightarrow{\omega} & B \vee B & \xrightarrow{\nabla} & B \end{array}$$

In some situations, even when the map  $F \xrightarrow{i} E$  is not null homotopic, there is a lift of the universal Whitehead product  $\nabla\omega$ . Suppose we are given a principal fibration defined by a map  $\varphi: B \rightarrow X$  where  $X$  is an  $H$ -space with strict unit:

$$\begin{array}{ccc} F & \xrightarrow{\cong} & \Omega X \\ \downarrow & & \downarrow \\ E & \xrightarrow{\quad} & PX \\ \downarrow \pi & & \downarrow \\ B & \xrightarrow{\varphi} & X. \end{array}$$

Here  $PX = \{\omega \in X^I \mid \omega(0) = *\}$  and  $E = \{(b, \omega) \in B \times PX \mid \omega(1) = \varphi(b)\}$ . This situation is also considered by Neisendorfer ([Nei10a]) where the author describes  $H$ -space based Samelson and Whitehead products. More generally, we will suppose that we are given maps  $x: K \rightarrow B$ ,  $y: L \rightarrow B$ . We then construct a strictly commuting diagram

$$(2.5) \quad \begin{array}{ccc} \Omega K * \Omega L & \xrightarrow{\Gamma} & E \\ \omega \downarrow & & \downarrow \pi \\ K \vee L & \xrightarrow{x \vee y} & B \end{array}$$

which is induced by the strictly commuting square

$$\begin{array}{ccc} K \vee L & \xrightarrow{x \vee y} & B \\ \downarrow & & \downarrow \varphi \\ K \times L & \xrightarrow{x \times y} X \times X \xrightarrow{\mu} & X \end{array}$$

where  $\mu$  is the  $H$ -space structure map. If  $K = L = B$  and  $x \vee y = \nabla$ , the map  $\Gamma$  is a lift of the universal Whitehead product to  $E$ .

We require an explicit formula for  $\Gamma$ . We first describe an equivalence of pairs

$$\xi: (C\Omega K, \Omega K) \rightarrow (PK, \Omega K)$$

by  $\xi(\omega, t)(s) = \omega(st)$ . Using  $\xi$ , we define an equivalence

$$\Omega K * \Omega L = (C\Omega K) \times \Omega L \cup \Omega K \times (C\Omega L) \xrightarrow{\xi^*} PK \times \Omega L \cup \Omega K \times PL$$

where both unions intersect in  $\Omega K \times \Omega L$ . The space  $PK \times \Omega L \cup \Omega K \times PL$  is the actual homotopy fiber of the map  $K \vee L \rightarrow K \times L$  and we define

$$\Gamma': PK \times \Omega L \cup \Omega K \times PL \rightarrow E$$

as follows: Write  $\Gamma'(\omega_1, \omega_2) = (\rho, \tilde{\omega}) \in B \times PX$  where

$$\rho = \begin{cases} x(\omega_1(t)) & \text{if } \omega_2(1) = * \\ y(\omega_2(t)) & \text{if } \omega_1(1) = * \end{cases}$$

$$\tilde{\omega}(t) = \mu(\varphi(x(\omega_1(t))), \varphi(y(\omega_2(t)))) .$$

Then  $\Gamma = \Gamma'\xi^*: \Omega K * \Omega L \rightarrow E$  and  $\pi\Gamma' = \rho$ , which is the composition

$$PK \times \Omega L \cup \Omega K \times PL \xrightarrow{e\nu} K \vee L \xrightarrow{x \vee y} B$$

where  $e\nu(\omega_1, \omega_2) = (\omega_1(1), \omega_2(1))$

**Proposition 2.6.** *Suppose that  $X$  is homotopy commutative. Then the maps*

$$\begin{array}{ccc} \Omega K * \Omega L & \xrightarrow{\Gamma_1} & E \\ \downarrow & & \downarrow \\ K \vee L & \xrightarrow{x \vee y} & B \end{array} \qquad \begin{array}{ccc} \Omega L * \Omega K & \xrightarrow{\Gamma_2} & E \\ \downarrow & & \downarrow \pi \\ L \vee K & \xrightarrow{y \vee x} & B \end{array}$$

are related by  $\Gamma_1 \sim \Gamma_2\tau$  where  $\tau: \Omega K * \Omega L \rightarrow \Omega L * \Omega K$  is given by the interchange  $PK * PL \leftrightarrow PL * PK$ .

*Proof.* Given  $\Gamma_1 = (\rho_1, \tilde{\omega}_1)$  and  $\Gamma_2 = (\rho_2, \tilde{\omega}_2)$  we see that  $\rho_1 = \rho_2$ . If we choose a commuting homotopy between  $\mu$  and  $\mu\tau$ ,  $H: X \times X \times I \rightarrow X$  which is constant on the axis  $(X \vee X) \times I$ , we get a homotopy from  $\tilde{\omega}_1$  to  $\tilde{\omega}_2$  which is constant at  $\tilde{\omega}(1) = \tilde{\omega}_2(1)$ . This defines a homotopy from  $\Gamma_1$  to  $\Gamma_2\tau$ .  $\square$

**Note 2.7.** For most considerations we will replace the join  $A * B$  with the suspended smash product  $\Sigma(A \wedge B)$  using the standard equivalence. Under this equivalence, the twist  $\Omega K * \Omega L \xrightarrow{\tau} \Omega L * \Omega K$  corresponds to the map

$$-\Sigma\tau: \Sigma(\Omega K \wedge \Omega L) \rightarrow \Sigma(\Omega L \wedge \Omega K)$$

where  $\tau$  is the interchange.

We apply this discussion to the construction of the Anick spaces. Since  $S^{2n+1}\{p^r\}$  has a homotopy commutative multiplication ([Nei80]), we obtain a map  $\Gamma_k$  in the diagram:

$$\begin{array}{ccccc} & & \Omega S^{2n+1}\{p^r\} & \xlongequal{\quad} & \Omega S^{2n+1}\{p^r\} \\ & & \downarrow & & \downarrow \\ \Omega G_k * \Omega G_k & \xrightarrow{\Gamma_k} & E_k & \longrightarrow & PS^{2n+1}\{p^r\} \\ \omega \downarrow & & \downarrow & & \downarrow \\ G_k \vee G_k & \xrightarrow{\nabla} & G_k & \xrightarrow{\varphi_k} & S^{2n+1}\{p^r\}. \end{array}$$

The construction of the fibration

$$T \longrightarrow R_k \longrightarrow G_k$$

depended on the choice of a map  $\nu_k: E_k \rightarrow BW_n$ . Suppose we can choose  $\nu_k$  such that the composition

$$\Omega G_k * \Omega G_k \xrightarrow{\Gamma_k} E_k \xrightarrow{\nu_k} BW_k$$

is null homotopic. Then we immediately get a lifting of  $\nabla\omega$  to  $R_k$ :

$$\begin{array}{ccccc} T & \longrightarrow & \Omega S^{2n+1}\{p^r\} & & \\ \downarrow & & \downarrow & & \\ & & R_k & \longrightarrow & E_k \xrightarrow{\nu_k} BW_n \\ & \nearrow \nabla\omega & \downarrow & \nearrow \Gamma_k & \downarrow \\ \Omega G_k * \Omega G_k & \xrightarrow{\quad} & G_k & \longrightarrow & G_k. \end{array}$$

We then have

**Proposition 2.7.** If we can choose a retraction  $\nu_k: E_k \rightarrow BW_n$  such that the composition

$$\Omega G_k * \Omega G_k \xrightarrow{\Gamma_k} E_k \xrightarrow{\nu_k} BW_n$$

is null homotopic for each  $k \geq 0$ , the corresponding fibration

$$\Omega G \longrightarrow T \longrightarrow R \longrightarrow G$$

will determine an Abelian  $H$ -space structure on  $T$ .

*Proof.* Since  $\nu_k$  is a retraction,  $\Omega E_k \simeq \Omega R_k \times W_n$ . Since the  $\Gamma_k$  are compatible for various  $k$ , it follows that the liftings  $\Omega G_k * \Omega G_k \rightarrow R_k$  are compatible as well. In the limit we have  $\Omega G * \Omega G \rightarrow R$  as required by 2.3.  $\square$

**Definition 2.8.** The level  $k - 1$  inductive assumption is that we have constructed a retraction  $\nu_{k-1}: E_{k-1} \rightarrow BW_n$  such that the composition  $\nu_k \Gamma_k$  is null homotopic.

We will assume the level  $k - 1$  inductive assumption throughout this work and succeed in proving the statement at level  $k$  in section 10. The level 0 assumption will be proved in section 8.

### 3. WHITEHEAD PRODUCTS

In this section we will recall the results of [Gra11] on generalized Whitehead products based on co- $H$  spaces. In addition we will discuss relative Whitehead products and  $H$ -space based Whitehead products in the total space of a principal fibration, mildly generalizing the results of Neisendorfer [Nei10a]. We will assume throughout this section that all spaces are simply connected.

Given two co- $H$  spaces  $G, H$ , we introduce a new<sup>4</sup> co- $H$  space  $G \circ H$  together with a cofibration sequence:

$$G \circ H \xrightarrow{W} G \vee H \longrightarrow G \times H.$$

To do this, suppose that  $G$  and  $H$  are given co- $H$  space structures by constructing right inverses to the respective evaluation maps:

$$\begin{aligned} G &\xrightarrow{\nu_1} \Sigma \Omega G \xrightarrow{\epsilon_1} G \\ H &\xrightarrow{\nu_2} \Sigma \Omega H \xrightarrow{\epsilon_2} H. \end{aligned}$$

We define a self map  $e: \Sigma(\Omega G \wedge \Omega H) \rightarrow \Sigma(\Omega G \wedge \Omega H)$  as the composition

$$\Sigma(\Omega G \wedge \Omega H) \xrightarrow{\epsilon_1 \wedge 1} G \wedge \Omega H \xrightarrow{\nu_1 \wedge 1} \Sigma(\Omega G \wedge \Omega H) \xrightarrow{1 \wedge \epsilon_2} \Omega G \wedge H \xrightarrow{1 \wedge \nu_2} \Sigma(\Omega G \wedge \Omega H);$$

$G \circ H$  is then defined as the telescopic direct limit of  $e$ . We then have:

**Proposition 3.1** ([Gra11, 2.1,2.3]). *The identity map of  $G \circ H$  factors:*

$$G \circ H \xrightarrow{\psi} \Sigma(\Omega G \wedge \Omega H) \xrightarrow{\theta} G \circ H.$$

---

<sup>4</sup>The discovery of the functor  $G \circ H$  was inspired by a result of Theriault [The03] where it was shown that the smash product of two simply connected co-associative co- $H$  spaces is the suspension of a co- $H$  space.

Furthermore, if  $f: G \rightarrow G'$  and  $g: H \rightarrow H'$  are co- $H$  maps, there are induced co- $H$  maps so that the diagram

$$\begin{array}{ccccc} G \circ H & \xrightarrow{\psi} & \Sigma(\Omega G \wedge \Omega H) & \xrightarrow{\theta} & G \circ H \\ f \circ g \downarrow & & \Sigma(\Omega f \wedge \Omega g) \downarrow & & f \circ g \downarrow \\ G' \circ H' & \xrightarrow{\psi'} & \Sigma(\Omega G' \wedge \Omega H') & \xrightarrow{\theta'} & G' \circ H' \end{array}$$

commutes up to homotopy.

Since  $G \circ H$  is the limit of the telescope defined by  $e$ ,  $\theta e \sim \theta$ , so the composition

$$G \circ H \xrightarrow{\psi} \Sigma(\Omega G \wedge \Omega H) \xrightarrow{e} \Sigma(\Omega G \wedge \Omega H) \xrightarrow{\theta} G \circ H$$

is homotopic to the identity. The map  $e$ , however, is a composition of 4 maps between co- $H$  spaces, and thus  $G \circ H$  is a retract of 3 different co- $H$  spaces and one of them,  $\Sigma(\Omega G \wedge \Omega H)$ , in two potentially distinct ways. This provides 4 potentially distinct co- $H$  space structures on  $G \circ H$ . We choose the structure defined by  $\psi$  and  $\theta$ ; viz.,

$$G \circ H \xrightarrow{\psi} \Sigma(\Omega G \wedge \Omega H) \xrightarrow{\Sigma \tilde{\theta}} \Sigma \Omega(G \circ H)$$

or equivalently

$$\begin{aligned} G \circ H &\xrightarrow{\psi} \Sigma(\Omega G \wedge \Omega H) \longrightarrow \Sigma(\Omega G \wedge \Omega H) \vee \Sigma(\Omega G \wedge \Omega H) \\ &\xrightarrow{\theta \wedge \theta} G \circ H \vee G \circ H \end{aligned}$$

where  $\tilde{\theta}$  is the adjoint of  $\theta$ .

**Proposition 3.2** ([Gra11, 2.3,2.5]). *There are co- $H$  equivalences  $G \circ \Sigma X \simeq G \wedge X$ ,  $\Sigma(G \circ H) \simeq G \wedge H$  which are natural for co- $H$  maps in  $G$  and  $H$  and continuous maps in  $X$ .*

**Proposition 3.3.** *There is a natural cofibration sequence*

$$G \circ H \xrightarrow{W} G \vee H \longrightarrow G \times H$$

where  $W$  is the composition:

$$G \circ H \xrightarrow{\psi} \Sigma(\Omega G \wedge \Omega H) \xrightarrow{\omega} G \vee H.$$

**Definition 3.4.** Let  $\alpha: G \rightarrow X$  and  $\beta: H \rightarrow X$ . We define the Whitehead product<sup>5</sup>

$$\{\alpha, \beta\}: G \circ H \rightarrow X$$

as the composition

$$G \circ H \xrightarrow{W} G \vee H \xrightarrow{\alpha \vee \beta} X.$$

**Proposition 3.5.** *Each Whitehead product  $\{\alpha, \beta\}: G \circ H \rightarrow X$  factors through the “universal Whitehead product”*

$$w = \nabla\omega: \Sigma(\Omega X \wedge \Omega X) \rightarrow X \vee X \rightarrow X.$$

*Proof.* The righthand square in the following diagram determines a map from the upper row which is a cofibration sequence to the lower row which is a fibration sequence

$$\begin{array}{ccccc} G \circ H & \xrightarrow{W} & G \vee H & \longrightarrow & G \times H \\ \xi \downarrow & & \alpha \vee \beta \downarrow & & \downarrow \\ \Sigma(\Omega X \wedge \Omega X) & \xrightarrow{\omega} & X \vee X & \longrightarrow & X \times X \end{array}$$

and  $\xi$  is unique up to homotopy since  $G \circ H$  is a co- $H$  space and  $\Omega(X \vee X) \simeq \Omega(X \times X) \times \Omega\Sigma(\Omega X \wedge \Omega X)$ . It follows that

$$\{\alpha, \beta\} = \nabla(\alpha \vee \beta)W \sim \nabla\omega\xi = w\xi. \quad \square$$

For the remainder of this section we will discuss  $H$ -space based Whitehead products and relative Whitehead products. In the case that  $G$  and  $H$  are Moore spaces, this material is covered in [Nei10a], and what we present is a mild generalization. We consider Whitehead products instead of their adjoints—the Samelson products. We also consider principal fibrations, so these products occur in the total space rather than the fiber. We wish to thank Joe Neisendorfer for several interesting conversations during the development of this material.

We begin with a principal fibration

$$\Omega X \xrightarrow{i} E \xrightarrow{\pi} B$$

classified by a map  $\varphi: B \rightarrow X$ . The (external) relative Whitehead product then is a pairing

$$[G, B] \times [H, E] \rightarrow [G \circ H, E].$$

---

<sup>5</sup>We use the notation  $\{\alpha, \beta\}$  rather than the usual  $[\alpha, \beta]$  since in an important application we need to make a distinction. That is the case when  $G$  and  $H$  are both Moore spaces. In this case  $G \circ H$  is a wedge of two Moore spaces. By choosing the higher dimensional one, Neisendorfer [Nei80] defines internal Whitehead products in homotopy with coefficients in  $Z/p^r$ . This is denoted  $[\alpha, \beta]$ , while  $\{\alpha, \beta\}$  is the “external” Whitehead product.

In case that  $X$  is an  $H$ -space with strict multiplication,<sup>6</sup> we also define the  $H$ -space based Whitehead product. It is a pairing

$$[G, B] \times [H, B] \rightarrow [G \circ H, E].$$

Suppose we are given maps:

$$G \xrightarrow{\alpha} B, H \xrightarrow{\beta} B, G \xrightarrow{\gamma} E, H \xrightarrow{\delta} E.$$

We will use the notation

$$\{\alpha, \gamma\}_r \in [G \circ H, E]$$

for the relative Whitehead product and

$$\{\alpha, \beta\}_\times \in [G \circ H, E]$$

for the  $H$ -space based Whitehead product. These products and the absolute Whitehead product are related by the following formulas to be proved:

$$(3.6c) \quad \pi\{\alpha, \beta\}_\times \sim \{\alpha, \beta\}: G \circ H \rightarrow B;$$

$$(3.6e) \quad \{\pi\gamma, \pi\delta\}_\times \sim \{\gamma, \delta\}: G \circ H \rightarrow E;$$

$$(3.10) \quad \{\alpha, \delta\}_r \sim \{\alpha, \pi\delta\}_\times: G \circ H \rightarrow E;$$

$$(3.11c) \quad \pi\{\alpha, \delta\}_r \sim \{\alpha, \pi\delta\}: G \circ H \rightarrow B;$$

$$(3.11e) \quad \{\pi\gamma, \delta\}_r \sim \{\gamma, \delta\}: G \circ H \rightarrow E.$$

We begin with the  $H$ -space based Whitehead product. These are defined using the map  $\Gamma$  from (2.5). We will shortly see that they are closely connected to the issues raised in section 2.

The product  $\{\alpha, \beta\}_\times$  is defined as the homotopy class of the composition:

$$\begin{array}{ccccc} G \circ H & \xrightarrow{\psi} & \Sigma(\Omega G \wedge \Omega H) & \xrightarrow{\Gamma} & E \\ & \searrow W & \downarrow \omega & & \downarrow \pi \\ & & G \vee H & \xrightarrow{\alpha \vee \beta} & B. \end{array}$$

**Proposition 3.6.** *Given  $\alpha: G \rightarrow B$  and  $\beta: H \rightarrow B$ , the homotopy class of the  $H$ -space based Whitehead product*

$$\{\alpha, \beta\}_\times: G \circ H \rightarrow E$$

*depends only on the homotopy classes of  $\alpha$  and  $\beta$ . Furthermore*

(a) *If  $f: G' \rightarrow G$  and  $g: H' \rightarrow H$  are co- $H$  maps,*

$$\{\alpha, \beta\}_\times(f \circ g) \sim \{\alpha f, \beta g\}_\times.$$

---

<sup>6</sup>Neisendorfer also assumes that the  $H$ -space structure is homotopy commutative. This is necessary to prove anticommutativity.

(b) *Given an induced fibration*

$$\begin{array}{ccc} E' & \xrightarrow{\tilde{\xi}} & E \\ \downarrow & & \downarrow \\ B' & \xrightarrow{\xi} & B \xrightarrow{\varphi} X \end{array}$$

and  $\alpha': G \rightarrow B'$ ,  $\beta': H \rightarrow B'$ , we have

$$\tilde{\xi}\{\alpha', \beta'\}_\times \sim \{\xi\alpha', \xi\beta'\}_\times : G \circ H \rightarrow E.$$

(c)  $\pi\{\alpha, \beta\}_\times \sim \{\alpha, \beta\}: G \circ H \rightarrow B$ .

(d) *Suppose  $\eta: X \rightarrow X'$  is a strict  $H$ -map and we have a pointwise commutative diagram*

$$\begin{array}{ccc} B & \xrightarrow{\xi} & B' \\ \varphi \downarrow & & \downarrow \varphi' \\ X & \xrightarrow{\eta} & X' \end{array}$$

which defines a map of principal fibrations:

$$\begin{array}{ccc} E & \xrightarrow{\tilde{\xi}} & E' \\ \downarrow & & \downarrow \\ B & \xrightarrow{\xi} & B'. \end{array}$$

Then

$$\tilde{\xi}\{\alpha, \beta\}_\times \sim \{\xi\alpha, \xi\beta\}_\times : G \circ H \rightarrow E'.$$

(e)  $\{\pi\gamma, \pi\delta\}_\times \sim \{\gamma, \delta\}: G \circ H \rightarrow E$ .

*Proof.* These all follow directly from the definition except for (e). To prove this we apply (d) to the diagram

$$\begin{array}{ccc} E & \xrightarrow{\pi} & B \\ \zeta \downarrow & & \downarrow \varphi \\ PX & \xrightarrow{ev} & X \end{array}$$

where  $\zeta(b, \omega) = \omega$ . Give  $PX$  the  $H$ -space structure of pointwise multiplication of paths in  $X$ . Then  $ev$  is a strict  $H$ -map. This gives a map of principal fibrations:

$$\begin{array}{ccc} \tilde{E} & \xrightarrow{\tilde{\pi}} & E \\ e \downarrow & & \downarrow \pi \\ E & \xrightarrow{\pi} & B. \end{array}$$

By (d) we have

$$\tilde{\pi}\{\gamma, \delta\}_\times \sim \{\pi\gamma, \pi\delta\}_\times.$$



However

$$\widetilde{E} = \{(b, \sigma) \in B \times X^{I \times I} \mid \varphi(b) = \sigma(0, 0), \sigma(s, 1) = \sigma(1, t) = *\}$$

and the maps  $\widetilde{\pi}$  and  $e$  are obtained by restriction of  $X^{I \times I}$  to  $X^{I \times 0}$  and  $X^{0 \times I}$ . These maps are homotopic by a  $90^\circ$  rotation in the square. Thus

$$\widetilde{\pi}\{\gamma, \delta\}_\times \sim e\{\gamma, \delta\}_\times \sim \{\gamma, \delta\}. \quad \square$$

We will describe the map  $\Gamma: \Sigma(\Omega G \wedge \Omega H) \rightarrow E$  in terms of iterated Whitehead products. Define  $G^{[i]} = G \circ G^{[i-1]}$  for  $i > 1$  and  $G^{[i]}H^{[j]}$  inductively by the formulas:

$$G^{[i]}H^{[j]} = \begin{cases} G \circ H^{[j]} & \text{if } i = 1 \\ G \circ (G^{[i-1]}H^{[j]}) & \text{if } i > 1 \end{cases}$$

Define  $ad^{i,j}: G^{[i]}H^{[j]} \rightarrow G \vee H$  inductively as the composition

$$G^{[i]}H^{[j]} \xrightarrow{W} G \vee G^{[i-1]}H^{[j]} \xrightarrow{1 \vee ad^{i-1,j}} G \vee G \vee H \xrightarrow{\nabla \vee 1} G \vee H$$

where  $G^{[0]}H^{[j]} = H^{[j]}$  and  $ad^{0,j}: H^{[j]} \rightarrow G \vee H$  is  $ad^j: H^{[j]} \rightarrow H$  included in  $G \vee H$ . In case  $i, j \geq 1$  these maps are null homotopic after inclusion into  $G \times H$ , so they factor uniquely through  $\Sigma(\Omega G \wedge \Omega H)$ :

$$\widetilde{ad}_\times^{i,j}: G^{[i]}H^{[j]} \rightarrow \Sigma(\Omega G \wedge \Omega H);$$

**Theorem 3.7** ([Gra11, 3b]). *The maps  $\widetilde{ad}_\times^{i,j}$  define a homotopy equivalence*

$$\bigvee_{\substack{i \geq 1 \\ j \geq 1}} G^{[i]}H^{[j]} \rightarrow \Sigma(\Omega G \wedge \Omega H)$$

*which is natural with respect to co- $H$  maps in either variable.*

**Theorem 3.8.** *Suppose*

$$\Omega X \longrightarrow E \longrightarrow G$$

*is a principal fibration classified by a map  $\varphi: G \rightarrow X$  where  $X$  is a homotopy commutative  $H$ -space with a strict unit and  $G$  is a co- $H$  space. Let  $\nu: E \rightarrow Z$ . Then the composition*

$$\Sigma(\Omega G \wedge \Omega G) \xrightarrow{\Gamma} E \xrightarrow{\nu} Z$$

*is null homotopic iff for each  $i \geq 2$  the composition*

$$G^{[i]} \longrightarrow \Sigma(\Omega G \wedge \Omega G) \xrightarrow{\Gamma} E \xrightarrow{\nu} Z$$

*is null homotopic.*

*Proof.* By 3.7 it suffices to show that for each  $i, j \geq 1$

$$G^{[i]}G^{[j]} \longrightarrow \Sigma(\Omega G \wedge \Omega G) \xrightarrow{\Gamma} E \xrightarrow{\nu} Z$$

is null homotopic, where  $i$  copies of  $G$  lie over the first factor of  $G \vee G$  and  $j$  copies over the second.

By 2.6,  $\Gamma\tau$  is homotopic to  $-\Gamma$ , where  $\tau: \Sigma(\Omega G \wedge \Omega G) \rightarrow \Sigma(\Omega G \wedge \Omega G)$  is the interchange of the two copies of  $\Omega G$ . It follows that the composition

$$G^{[i]}G^{[j]} \xrightarrow{\widetilde{ad}_\times^{i,j}} \Sigma(\Omega G \wedge \Omega G) \xrightarrow{\Gamma} E$$

depends only on  $i + j$ , up to sign. Since  $G^{[i]}G^{[j]} = G^{[i+j]}$  the result follows.  $\square$

This will be applied to 2.7 in the sequel.

We now turn to consideration of the relative Whitehead product. As before we are working with a principal fibration

$$\Omega X \xrightarrow{i} E \xrightarrow{\pi} B$$

induced by a map  $\varphi: B \rightarrow X$ , so

$$E = \{(b, \omega) \in B \times PX \mid \varphi(b) = \omega(1), * = \omega(0)\}.$$

We no longer need to assume that  $X$  is an  $H$ -space. We have a strictly commutative square

$$\begin{array}{ccc} B \vee E & \xrightarrow{1 \vee \pi} & B \\ \downarrow & & \downarrow \varphi \\ B \vee CE & \xrightarrow{\varphi \vee K} & X \end{array}$$

where  $K(b, \omega, s) = \omega(s)$ .  $K$  is the canonical null homotopy of the composition:

$$E \xrightarrow{\pi} B \xrightarrow{\varphi} X_j$$

Let  $F_1$  be the homotopy fiber of the inclusion  $B \vee E \rightarrow B \vee CE$  and

$$\Gamma_1: F_1 \rightarrow E$$

the induced maps on homotopy fibers. Since the map  $B \vee E \rightarrow B \vee CE$  has a right homotopy inverse,

$$\Omega(B \vee E) \simeq F_1 \times \Omega(B \vee CE).$$

Given maps  $\alpha: G \rightarrow B$  and  $\delta: H \rightarrow E$ , the Whitehead product  $\{\alpha, \delta\}: G \circ H \rightarrow B \vee E$  projects trivially to  $B \vee CE$  and there is consequently a lifting to  $F_1$  which is unique up to homotopy:

$$\begin{array}{ccccccc} & & & & F_1 & \xrightarrow{\Gamma_1} & E \\ & & & \nearrow \theta & \downarrow & & \downarrow \\ G \circ H & \xrightarrow{\quad} & G \vee H & \xrightarrow{\alpha \vee \delta} & B \vee E & \xrightarrow{1 \vee \pi} & B. \end{array}$$

**Definition 3.9.** The relative Whitehead product

$$\{\alpha, \delta\}_r: G \circ H \rightarrow E$$

is the homotopy class of the composition  $\Gamma_1 \cdot \theta$ .

We seek to compare the relative Whitehead product and the  $H$ -space based Whitehead product when they are both defined. Recall that the  $H$ -space based Whitehead product is similarly defined as a map between induced fibrations:

$$\Sigma(\Omega B \wedge \Omega B) \simeq \begin{array}{ccc} F_2 & \xrightarrow{\Gamma} & E \\ \downarrow & & \downarrow \\ B \vee B & \xrightarrow{\nabla} & B \\ \downarrow & & \downarrow \varphi \\ B \times B & \xrightarrow{\varphi \times \varphi} X \times X \xrightarrow{\mu} & X. \end{array}$$

We also define a mixed product when  $X$  has a fixed  $H$ -space structure map  $\mu$ :

$$\begin{array}{ccc} F_3 & \xrightarrow{\Gamma_3} & E \\ \downarrow & & \downarrow \\ B \vee E & \xrightarrow{1 \vee \pi} & B \\ \downarrow & & \downarrow \\ B \times CE & \xrightarrow{\varphi \times K} X \times X \xrightarrow{\mu} & X. \end{array}$$

It is clear that  $\Gamma_1$  factors through  $\Gamma_3$

$$\begin{array}{ccccc} F_1 & \xrightarrow{\simeq} & F_3 & \xrightarrow{\Gamma_3} & E \\ \downarrow & & \downarrow & & \downarrow \\ B \vee E & \xlongequal{\quad} & B \vee E & \longrightarrow & B \\ \downarrow & & \downarrow & & \downarrow \\ B \vee CE & \longrightarrow & B \times CE & \longrightarrow & X \end{array}$$

and the induced map  $F_1 \rightarrow F_3$  is a homotopy equivalence. Likewise, the  $H$ -space based product  $\{\alpha, \pi\}_\times$  is represented by the upper composition in the diagram

$$\begin{array}{ccccc} F_4 & \longrightarrow & F_2 & \xrightarrow{\Gamma} & E \\ \downarrow & & \downarrow & & \downarrow \\ B \vee E & \xrightarrow{1 \vee \pi} & B \vee B & \xrightarrow{\nabla} & B \\ \downarrow & & \downarrow & & \downarrow \varphi \\ B \times E & \xrightarrow{1 \vee \pi} & B \times B & \xrightarrow{\varphi \times \varphi} X \times X \xrightarrow{\mu} & X \end{array}$$

which also factors through  $F_3$ .

Since  $\Omega(B \vee E) \simeq \Omega F_4 \times \Omega(B \times E)$ , the Whitehead product

$$\{\alpha, \delta\}: G \circ H \rightarrow B \vee E$$

has a unique left to  $F_4$  and the image in  $E$  is both homotopic to  $\{\alpha, \delta\}_r$  and  $\{\alpha, \pi\delta\}_\times$ . We have proved

**Proposition 3.10.** *Suppose  $\alpha: G \rightarrow B$ , and  $\delta: H \rightarrow E$ , and  $X$  is an  $H$ -space with strict unit. Then*

$$\{\alpha, \delta\}_r \sim \{\alpha, \pi\delta\}_\times \quad \square$$

Analogous to 3.6, we have

**Proposition 3.11.** *The homotopy class of the relative Whitehead product  $\{\alpha, \delta\}_r$  depends only on the homotopy classes of  $\alpha$  and  $\delta$ . Furthermore*

(a) *If  $f: G' \rightarrow G$  and  $g: H' \rightarrow H$  are co- $H$  maps, then*

$$\{\alpha, \delta\}_r \cdot (f \circ g) \sim \{\alpha f, \delta g\}_r.$$

(b) *Given an induced fibration*

$$\begin{array}{ccc} E' & \xrightarrow{\tilde{\xi}} & E \\ \downarrow & & \downarrow \\ B' & \xrightarrow{\xi} & B \xrightarrow{\varphi} X \end{array}$$

and classes  $\alpha': G \rightarrow B'$ ,  $\delta': H \rightarrow E'$ , we have

$$\tilde{\xi}\{\alpha', \delta'\}_r \sim \{\xi\alpha', \tilde{\xi}\delta'\}_r.$$

(c)  $\pi\{\alpha, \delta\}_r \sim \{\alpha, \pi\delta\}$ .

(d) *Suppose  $\eta: X \rightarrow X'$  induces a map of induced fibrations:*

$$\begin{array}{ccc} E & \xrightarrow{\bar{\eta}} & E' \\ \downarrow & & \downarrow \\ B & \xlongequal{\quad} & B \\ \downarrow & & \downarrow \\ X & \xrightarrow{\eta} & X'. \end{array}$$

Then  $\bar{\eta}\{\alpha, \delta\}_r \sim \{\alpha, \bar{\eta}\delta\}_r$

(e) *if  $\gamma: G \rightarrow E$ , then*

$$\{\pi\gamma, \delta\}_r \sim \{\gamma, \delta\}.$$

*Proof.* All except part (e) follow directly from the definition. Part (e) is handled as in 3.6 by the pullback over  $\pi$ .  $\square$

Whenever possible, we will replace  $H$ -space based Whitehead products with relative Whitehead products. This is because we have better control over the homological properties of these products. At this point we will develop a formula to do that. First we need to have a more precise construction of the map  $\Gamma'$ .

As in the case of  $\Gamma$  in 2.6, we describe the homotopy fiber  $F_0$  of the projection map

$$B \vee E \xrightarrow{\pi_1} B$$

as  $PB \cup_{\Omega B} \Omega B \times E$ . Using the equivalence  $\xi: (C\Omega X, \Omega X) \rightarrow (PX, \Omega x)$  from section 2, we see that this space is homotopy equivalent to

$$C\Omega B \cup_{\Omega B} \Omega B \times E \simeq \Omega B \ltimes E$$

and the map  $\Gamma': C\Omega B \cup_{\Omega B} \Omega B \times E \rightarrow E$  is described as follows:

The restriction of  $\Gamma'$  to  $\Omega B \times E$  is the composition

$$\Omega B \times E \xrightarrow{\Omega\varphi \times 1} \Omega X \times E \xrightarrow{a} E$$

where  $a: \Omega X \times E \rightarrow E$  is the principal action. The restriction of this map to  $\Omega B \times *$  is the null homotopic composition of maps in the fibration sequence:

$$\Omega B \rightarrow \Omega X \rightarrow E.$$

This defines the extension over  $C\Omega B$  and consequently:

$$\Omega B \ltimes E \simeq C\Omega B \cup \Omega B \times E \rightarrow E.$$

**Proposition 3.12.** *Suppose  $\alpha: SA \rightarrow B$  and  $\gamma: G \rightarrow E$ . Let  $\widetilde{\varphi}\alpha: A \rightarrow \Omega X$  be the adjoint of  $\varphi\alpha: SA \rightarrow X$  and  $a: \Omega X \times E \rightarrow E$  be the principal action. Then the induced map in homology*

$$(\{\alpha, \gamma\}_r)_*: H_*(A \wedge G) \rightarrow H_*(E)$$

is determined by the commutative diagram:

$$\begin{array}{ccccccc} H_*(A \wedge G) & \hookrightarrow & H_*(A \ltimes G) & \longrightarrow & H_*(\Omega B \ltimes E) & \xrightarrow{(\Gamma')_*} & H_*(E) \\ & & \uparrow & & \uparrow & & \uparrow a_* \\ & & H_*(A \times G) & \longrightarrow & H_*(\Omega B \times E) & \xrightarrow{(\Omega\varphi \times 1)_*} & H_*(\Omega X \times E). \end{array}$$

#### 4. CONSTRUCTION OF THE CO- $H$ LADDER

In this section we construct a cofibration ladder that clarifies the relationship between  $E_{k-1}$  and  $E_k$ . This ladder is a lifting of the one used to prove [GT10, 4.3(e)]. To do this we construct a sharper version of [GT10, 4.3(d)].

We make essential use of property B of section 2.

(B)  $\Sigma(T \wedge T)$  is a wedge of Moore spaces.

We assume some arbitrary choice of a map  $\nu_k$  has been made as in [GT10]. We have a corresponding  $H$ -space structure on  $T$  and easily calculate that

$$H_*(T_j Z/p) \simeq Z/p[v] \otimes \wedge(u)$$

where  $|v| = 2n$ ,  $|u| = 2n - 1$  and  $\beta^{(r)}v = u$  where  $\beta^{(r)}$  is the  $r^{\text{th}}$  homology Bockstein. Since  $G$  is a retract of  $\Sigma T$ , we conclude that  $\beta^{(r+i)}v^{p^i} = uv^{p^i-1}$ . Using the  $H$ -space structure map  $\mu$  we consider the Hopf construction:

$$H(\mu): \Sigma(T \wedge T) \rightarrow \Sigma T.$$

Note that in homology

$$(H(\mu))_*(\sigma \otimes x \otimes y) = \sigma \otimes \mu_*(x \otimes y)$$

if  $|x| > 0$  and  $|y| > 0$ . We now define homology classes

$$\begin{aligned} \alpha &\in H_{2np^k+1}(\Sigma(T \wedge T); \mathbb{Z}/p) \\ \beta &\in H_{2np^k}(\Sigma(T \wedge T); \mathbb{Z}/p) \end{aligned}$$

by the formulas

$$\begin{aligned} \alpha &= \sigma \otimes v^{p^{k-1}} \otimes v^{p^{k-1}(p-1)} \\ \beta &= \sigma \otimes v^{p^{k-1}} \otimes uv^{p^{k-1}(p-1)-1} \end{aligned}$$

so we have

$$\begin{aligned} (H(\mu))_* \alpha &= \sigma \otimes v^{p^k} \\ (H(\mu))_* \beta &= \sigma \otimes uv^{p^k-1}. \end{aligned}$$

Now  $\beta^{(r+k-1)}(\alpha)$  and  $\beta^{(r+k-1)}(\beta)$  are both nonzero. Since  $\Sigma(T \wedge T)$  is a wedge of Moore spaces, there are maps

$$\begin{aligned} a: P^{2np^k+1}(p^{r+k-1}) &\rightarrow \Sigma(T \wedge T) \\ b: P^{2np^k}(p^{r+k-1}) &\rightarrow \Sigma(T \wedge T) \end{aligned}$$

such that  $\alpha$  is in the image of  $a_*$  and  $\beta$  is in the image of  $b_*$ . Combining these we get a map  $e$

$$P^{2np^k+1}(p^{r+k-1}) \vee P^{2np^k}(p^{r+k-1}) \xrightarrow{a \vee b} \Sigma(T \wedge T) \xrightarrow{H(\mu)} \Sigma T$$

such that  $\sigma \otimes v^{p^k}$  and  $\sigma \otimes uv^{p^k-1}$  are in the image of  $e_*$ . We have proved

**Proposition 4.1.** *For any  $H$ -space structure map  $\mu: T \times T \rightarrow T$  there is a homotopy commutative diagram*

$$\begin{array}{ccc} P^{2np^k}(p^{r+k-1}) \vee P^{2np^k+1}(p^{r+k-1}) & \xrightarrow{e} & \Sigma T^{2np^k} \\ \downarrow b \vee a & & \downarrow \\ \Sigma(T \wedge T) & \xrightarrow{H(\mu)} & \Sigma T \end{array}$$

where  $e$  induces an epimorphism in mod  $p$  homology in dimensions  $2np^k$  and  $2np^k + 1$ .  $\square$

Now let  $\tilde{E}: \Sigma T \rightarrow S^{2n+1}\{p^r\}$  be the adjoint of the map  $E: \Gamma \rightarrow \Omega S^{2n+1}\{p^r\}$  from (2.2).

**Proposition 4.2.** *The composition*

$$\Sigma(T \wedge T) \xrightarrow{H(\mu)} \Sigma T \xrightarrow{\tilde{E}} S^{2n+1}\{p^r\}$$

*is null homotopic.*

*Proof.* By property  $E$  of section 2, the map  $E: T \rightarrow \Omega S^{2n+1}\{p^r\}$  is an  $H$  map. Consequently there is a homotopy commutative diagram:

$$\begin{array}{ccc} \Sigma(T \wedge T) & \xrightarrow{\Sigma(E \wedge E)} & \Sigma(\Omega S^{2n+1}\{p^r\} \wedge \Omega S^{2n+1}\{p^r\}) \\ H(\mu) \downarrow & & \downarrow H(\mu') \\ \Sigma T & \longrightarrow & \Sigma \Omega S^{2n+1}\{p^r\}. \end{array}$$

where  $\mu'$  is the loop space structure map on  $\Omega S^{2n+1}\{p^r\}$ . However, since  $\Omega S^{2n+1}\{p^r\}$  is a loop space, the righthand map is part of the classifying space structure

$$\begin{array}{ccc} \Sigma(\Omega S^{2n+1}\{p^r\} \wedge \Omega S^{2n+1}\{p^r\}) & \longrightarrow \dots \longrightarrow & E_\infty \\ H(\mu') \downarrow & & \downarrow \\ \Sigma \Omega S^{2n+1}\{p^r\} & \longrightarrow \dots \longrightarrow & S^{2n+1}\{p^r\} \end{array}$$

where  $E_\infty$  is contractible and the bottom horizontal map is the evaluation map. The result follows since  $\tilde{E}$  is the composition:

$$\Sigma T \longrightarrow \Sigma \Omega S^{2n+1}\{p^r\} \xrightarrow{ev} S^{2n+1}\{p^r\}. \quad \square$$

We now use (A) from section 2 to construct a map  $g_k$  in the diagram:

$$\begin{array}{ccccc} & & \Omega G_k & \xlongequal{\quad} & \Omega G_k \\ & & \downarrow & & \downarrow \Omega' \varphi_k \\ T^{2np^k} & \longrightarrow & T^{2np^{k+1}-2} & \xrightarrow{g_k} & T \\ & & \downarrow & & \downarrow \\ & & R_k & \longrightarrow & E_k. \end{array}$$

Taking adjoints and combining with 4.1, we construct a homotopy commutative diagram

$$\begin{array}{ccc} P^{2np^k}(p^{r+k-1}) \vee P^{2np^{k+1}}(p^{r+k-1}) & \longrightarrow & \Sigma(T \wedge T) \\ e \downarrow & & \downarrow H(\mu) \\ \Sigma T^{2np^k} & \longrightarrow & \Sigma T \\ \tilde{g}_k \downarrow & & \downarrow \tilde{E} \\ G_k & \xrightarrow{\varphi_k} & S^{2n+1}\{p^r\} \end{array}$$

in which the righthand vertical composition is null homotopic. This implies

**Proposition 4.3.** *For any  $H$ -space structure on  $T$  and any choice of a lifting  $g_k: T^{2np^k} \rightarrow \Omega G_k$  of the inclusion of  $T^{2np^k}$  into  $T$ , there is a lifting<sup>7</sup> of  $\tilde{g}_k e$  to  $E_k$*

$$\begin{array}{ccc} P^{2np^k}(p^{r+k-1}) \vee P^{2np^k+1}(p^{r+k-1}) & \xrightarrow{a(k) \vee c(k)} & E_k \\ e \downarrow & & \downarrow \pi_k \\ \Sigma T^{2np^k} & \xrightarrow{\tilde{g}_k} & G_k. \end{array}$$

In the diagram below, the left column is a standard cofibration sequence and the right column is a fibration sequence defined by the projection  $\pi$ :

$$\begin{array}{ccccc} P^{2np^k}(p^{r+k}) & \xrightarrow{\theta_1} & \Omega P^{2np^k+1}(p^{r+k}) & & \\ p^{r+k-1} \downarrow & & \downarrow & & \\ P^{2np^k}(p^{r+k}) & \xrightarrow{\theta_2} & J & & \\ \sigma \vee \sigma\beta \downarrow & & \downarrow & & \\ P^{2np^k}(p^{r+k-1}) \vee P^{2np^k+1}(p^{r+k-1}) & \xrightarrow{a(k) \vee c(k)} & E_k & \xrightarrow{\pi_k} & G_k \\ -\delta_1 \vee \rho \downarrow & & & & \downarrow \pi \\ P^{2np^k+1}(p^{r+k}) & \xlongequal{\quad\quad\quad} & P^{2np^k+1}(p^{r+k}). & & \end{array}$$

The homological properties of  $e$  and 4.3 imply that the bottom region commutes up to homotopy since  $\tilde{g}_k$  has a right homotopy inverse by (C) of section 2. The maps  $\theta_1$  and  $\theta_2$  are induced from this region in the standard way. For dimensional reasons  $\theta_2$  factors through  $G_{k-1} \subset J$  and since

$$\begin{array}{ccc} E_{k-1} & \longrightarrow & G_{k-1} \\ \downarrow & & \downarrow \\ E_k & \longrightarrow & G_k \end{array}$$

is a pullback diagram,  $\theta_2$  factors through  $E_{k-1}$ .  $\theta_1$  factors through

$$P^{2np^k-1}(p^{r+k})$$

also for dimensional reasons. We obtain

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<sup>7</sup>It will be seen later that we can choose  $\nu_k$  such that  $\nu_k(a(k) \vee c(k))$  is null homotopic and  $a(k) \vee c(k)$  are unique up to homotopy with this property.



**Theorem 4.4.** *There is a homotopy commutative ladder of fibrations:*

$$\begin{array}{ccccc}
 P^{2np^k}(p^{r+k}) & \xlongequal{\hspace{2cm}} & P^{2np^k}(p^{r+k}) & & \\
 \downarrow p^{r+k-1} & & & & \downarrow \alpha_k \\
 P^{2np^k}(p^{r+k}) & \xrightarrow{\beta_k} & E_{k-1} & \xrightarrow{\pi_{k-1}} & G_{k-1} \\
 \downarrow \sigma \vee \sigma\beta & & \downarrow & & \downarrow \\
 P^{2np^k}(p^{r+k-1}) \vee P^{2np^{k+1}}(p^{r+k-1}) & \xrightarrow{a(k) \vee c(k)} & E_k & \xrightarrow{\pi_k} & G_k \\
 \downarrow -\delta_1 \vee \rho & & & & \downarrow \\
 P^{2np^{k+1}}(p^{r+k}) & \xlongequal{\hspace{2cm}} & P^{2np^{k+1}}(p^{r+k}) & & 
 \end{array}$$

Furthermore, for any choice of retraction  $\nu_{k-1}: E_{k-1} \rightarrow BW_n$ , we can choose  $\beta_k$  so that  $\nu_{k-1}\beta_k \sim *$ , and alter  $a(k)$  and  $c(k)$  so that all diagrams still commute up to homotopy.

*Proof.* We need only demonstrate the last statement. Suppose we are given a map  $\bar{\beta}_k: P^{2np^k}(p^{r+k}) \rightarrow E_{k-1}$  so that the diagram commutes up to homotopy. Given a retraction  $\nu_{k-1}: E_{k-1} \rightarrow BW_n$ , we get a splitting

$$\Omega E_{k-1} \simeq \Omega R_{k-1} \times W_n$$

where  $R_{k-1}$  is the fiber of  $\nu_{k-1}$ . We can then write  $\beta_k = \bar{\beta}_k - \epsilon$  where  $\epsilon$  is the component of  $\bar{\beta}_k$  that factors through  $W_n$  and  $\beta_k$  factors through  $R_{k-1}$ . Since each map  $P^{2np^k}(p^{r+k}) \rightarrow W_n$  has order  $p$ ,  $\epsilon$  has order  $p$  and thus  $p^{r+k-1}\bar{\beta}_k = p^{r+k-1}\beta_k$  as  $r+k-1 \geq 1$ . Thus the upper region commutes up to homotopy when  $\bar{\beta}_k$  is replaced by  $\beta_k$ . Since  $p^{r+k-1}\epsilon = 0$ ,  $\epsilon$  factors

$$P^{2np^k}(p^{r+k}) \xrightarrow{\sigma \vee \sigma\beta} P^{2np^k}(p^{r+k-1}) \vee P^{2np^{k+1}}(p^{r+k-1}) \xrightarrow{\epsilon'} E_{k-1}$$

as the lefthand column is a cofibration sequence. We now redefine  $a(k)$  and  $c(k)$  by subtracting off the appropriate components of  $\epsilon'$  and the middle region now commutes up to homotopy. Since this alteration of  $a(k)$  and  $c(k)$  factors through  $E_{k-1}$ , the projections to  $G_k$  vanish when projected to  $P^{2np^k}(p^{r+k})$  so the bottom region also commutes up to homotopy.  $\square$

**Note 4.5.** *Given the inductive hypothesis at level  $k-1$ , we now assume that the alterations in 4.4 have been made so that in establishing the statement at level  $k$ , we have the diagram in 4.4 with  $\beta_k$  in the kernel of  $\nu_{k-1}$ .*

## 5. APPROXIMATION

The goal of this section is to replace the co- $H$  space  $G_k$  by a sequence of approximations. The end result will be to replace  $G_k^{[2]}$  by a wedge of mod  $p^s$  Moore spaces for  $s \geq r$ . We begin with a cofibration sequence based on the ladder 4.4. Throughout this section we will exclude the case  $n = 1$ .

**Proposition 5.1.** *For  $k \geq 1$  there is a cofibration sequence*

$$P^{2np^k}(p^{r+k}) \xrightarrow{\xi_k} L_k \xrightarrow{\zeta_k} G_k \xrightarrow{\pi'} P^{2np^k+1}(p^{r+k})$$

where  $L_k = G_{k-1} \vee P^{2np^k}(p^{r+k-1}) \vee P^{2np^k+1}(p^{r+k-1})$ ,  $\zeta_k$  is induced by the inclusion of  $G_{k-1}$  and the maps  $\pi_k a(k)$  and  $\pi_k c(k)$ , and  $\pi' = p^{r+k-1}\pi$ .

*Proof.* This is a standard consequence of a ladder in which each third rail is an equivalence.  $\square$

**Proposition 5.2.** *If  $n > 1$ , there is a unique co- $H$  space structure on  $L_k$  so that the cofibration in 5.1 is a cofibration of co- $H$  maps.*

*Proof.* Let  $P$  be the pullback in the diagram:

$$\begin{array}{ccc} P & \longrightarrow & G_k \vee G_k \\ \downarrow & & \downarrow \\ L_k \times L_k & \longrightarrow & G_k \times G_k. \end{array}$$

There is a map  $\eta: L_k \rightarrow P$  which projects to the diagonal map on  $L_k$  and the composition:

$$L_k \xrightarrow{\zeta_k} G_k \longrightarrow G_k \vee G_k.$$

We first assert that  $\eta$  is unique up to homotopy. Since  $L_k$  is a wedge of co- $H$  spaces, it suffices to show that if  $\epsilon: L_k \rightarrow P$  projects trivially to  $G_k \vee G_k$  and  $L_k \times L_k$ , it is itself trivial. Now the homotopy fiber of the map  $P \rightarrow L_k \times L_k$  is the same as the homotopy fiber of  $G_k \vee G_k \rightarrow G_k \times G_k$ , i.e.,  $\Sigma(\Omega G_k \wedge \Omega G_k)$ ; we conclude that  $\epsilon$  must factor through  $\Sigma(\Omega G_k \wedge \Omega G_k)$ , and that the composition

$$P \xrightarrow{\epsilon'} \Sigma(\Omega G_k \wedge \Omega G_k) \longrightarrow G_k \vee G_k$$

is null homotopic. Since  $P$  is a co- $H$  space, this implies that  $\epsilon'$  is null homotopic and hence  $\epsilon$  is as well since  $\Omega(G_k \vee G_k) \rightarrow \Omega(G_k \times G_k)$  has a right homotopy inverse.

The map  $\zeta_k$  is a  $2np^k - 1$  equivalence since  $P^{2np^k+1}(p^{r+k})$  is  $2np^k - 1$  connected. Since  $L_k$  and  $G_k$  are both  $2n - 1$  connected, this implies that the composition

$$\Sigma(\Omega L_k \wedge \Omega L_k) \longrightarrow \Sigma(\Omega L_k \wedge \Omega G_k) \longrightarrow \Sigma(\Omega G_k \wedge \Omega G_k)$$

is a  $2np^k + 2n - 2$  equivalence. Now consider the diagram of vertical fibrations:

$$\begin{array}{ccccc}
 \Sigma(\Omega L_k \wedge \Omega L_k) & \longrightarrow & \Sigma(\Omega G_k \wedge \Omega G_k) & \xrightarrow{\simeq} & \Sigma(\Omega G_k \wedge \Omega G_k) \\
 \downarrow & & \downarrow & & \downarrow \\
 L_k \vee L_k & \longrightarrow & P & \longrightarrow & G_k \vee G_k \\
 \downarrow & & \downarrow & & \downarrow \\
 L_k \times L_k & \xlongequal{\quad} & L_k \times L_k & \longrightarrow & G_k \times G_k.
 \end{array}$$

From this we see that the map  $L_k \vee L_k \rightarrow P$  is a  $2np^k + 2n - 2$  equivalence. Since  $n > 1$ , that implies that there is a unique lifting of  $\eta: L_k \rightarrow P$  to  $L_k \vee L_k$ , which defines a co- $H$  space structure on  $L_k$  such that  $\zeta_k$  is a co- $H$  map.

Similarly, we observe that the  $2np^k + 2n - 3$  skeleton of the fiber of the map  $L_k \vee L_k \rightarrow G_k \vee G_k$  is  $P^{2np^k}(p^{r+k}) \vee P^{2np^k}(p^{r+k})$ , so the composition

$$P^{2np^k}(p^{r+k}) \xrightarrow{\xi_k} L_k \longrightarrow L_k \vee L_k$$

factors through  $P^{2np^k}(p^{r+k}) \vee P^{2np^k}(p^{r+k})$  and such a factorization defines a co- $H$  space structure on  $P^{2np^k}(p^{r+k})$ . That structure, of course, is unique. So this  $\xi_k$  is a co- $H$  map with the suspension structure.  $\square$

It should be pointed out that  $L_k$  does not split as a co-product of co- $H$  spaces. In particular, the inclusion  $P^{2np^k+1}(p^{r+k-1}) \rightarrow L_k$  is not a co- $H$  map. If it were, the map

$$P^{2np^k+1}(p^{r+k-1}) \xrightarrow{c(k)} E_k \xrightarrow{\pi_k} G_k$$

would be a co- $H$  map, contradicting [AG95, 2.2].

Write  $[k]: \Sigma X \rightarrow \Sigma X$  for the  $k$ -fold sum of the identity map.

**Definition 5.3.** Suppose  $G \xrightarrow{f} H \xrightarrow{g} \Sigma K$  is a cofibration sequence of co- $H$  spaces and co- $H$  maps. We will say that  $f$  is an index  $p$  approximation if there is a co- $H$  map  $g': \Sigma H \rightarrow \Sigma^2 K$  such that  $\Sigma g$  factors

$$\Sigma H \xrightarrow{g'} \Sigma^2 K \xrightarrow{[p]} \Sigma^2 K$$

up to homotopy.  $f: G \rightarrow H$  will be called an iterated index  $p$  approximation if  $f$  is homotopic to a composition

$$G = G_0 \rightarrow G_1 \rightarrow \cdots \rightarrow G_m = H$$

where each map  $G_i \rightarrow G_{i+1}$  is an index  $p$  approximation.

Thus, for example,  $\zeta_k: L_k \rightarrow G_k$  is an index  $p$  approximation.

**Proposition 5.4.** *Suppose that  $f: G \rightarrow H$  is an iterated index  $p$  approximation and  $\nu: H \rightarrow BW_n$ . Then  $\nu$  is null homotopic iff  $\nu f$  is null homotopic.*

*Proof.* We will only consider the case when  $f$  is an index  $p$  approximation, as the general result follows by an easy induction. Suppose then that  $f: G \rightarrow H$  is an index  $p$  approximation and  $\Sigma g$  factors up to homotopy:

$$\Sigma H \xrightarrow{g'} \Sigma^2 K \xrightarrow{[p]} \Sigma^2 K.$$

Assume that  $\nu f$  is null homotopic, so we can factor  $\nu$  as

$$H \xrightarrow{g} \Sigma K \xrightarrow{\nu'} BW_n.$$

Consider the diagram:

$$\begin{array}{ccccccc} \Omega \Sigma^2 K & \xrightarrow{\Omega[p]} & \Omega \Sigma^2 K & \xrightarrow{\Omega \Sigma \nu'} & \Omega \Sigma BW_n & \longrightarrow & BW_n \\ \tilde{g}' \uparrow & & \uparrow & & \uparrow & \nearrow & \\ H & \xrightarrow{g} & \Sigma K & \xrightarrow{\nu'} & BW_n & & \end{array}$$

Since  $BW_n$  is homotopy associative ([Gra88]), the upper composition is an  $H$ -map. This composition is thus inessential if its restriction to  $\Sigma K$  is inessential. However this restriction factors through  $[p]: \Sigma K \rightarrow \Sigma K$ . Since  $BW_n$  has  $H$ -space exponent  $p$  ([The07]), we conclude that the upper composition is inessential. This  $\nu \sim \nu' g$  is inessential as well.  $\square$

**Lemma 5.5.** *There is a homotopy commutative diagram*

$$\begin{array}{ccc} (\Sigma^2 H) \circ K & \xrightarrow{[p] \circ 1} & (\Sigma^2 H) \circ K \\ \wr \downarrow & [p] \wedge 1 & \wr \downarrow \\ (\Sigma H) \wedge K & \xrightarrow{[p] \wedge 1} & (\Sigma H) \wedge K \\ \wr \downarrow & [p] & \wr \downarrow \\ \Sigma^2(H \circ K) & \xrightarrow{[p]} & \Sigma^2(H \circ K) \end{array}$$

where the equivalences are co- $H$  equivalences.

*Proof.* The vertical equivalences follow from 3.2. These equivalences are natural for co- $H$  maps. However  $[p]: \Sigma H \rightarrow \Sigma H$  is a co- $H$  map since  $H$  is a co- $H$  space.  $\square$

**Lemma 5.6.** *Suppose  $G_1 \xrightarrow{\alpha} G_2 \xrightarrow{\beta} G_3$  is a cofibration sequence of co- $H$  spaces and co- $H$  maps. Then for each co- $H$  space  $H$ ,*

$$\begin{array}{ccc} G_1 \circ H & \xrightarrow{\alpha \circ 1} & G_2 \circ H \xrightarrow{\beta \circ 1} G_3 \circ H \\ H \circ G_1 & \xrightarrow{1 \circ \alpha} & H \circ G_2 \xrightarrow{1 \circ \beta} H \circ G_3 \end{array}$$

are both cofibration sequences.

*Proof.* In the following sequence, the composition of two adjacent maps is null homotopic

$$G_1 \xrightarrow{\alpha} G_2 \xrightarrow{\beta} G_3 \xrightarrow{\gamma} \Sigma G_1 \xrightarrow{\Sigma\alpha} \Sigma G_2 \xrightarrow{\Sigma\beta} \Sigma G_3 \longrightarrow \dots$$

and all maps are co- $H$  maps. It follows that the same is true for the sequence:

$$G_1 \circ H \xrightarrow{\alpha \circ 1} G_2 \circ H \xrightarrow{\beta \circ 1} G_3 \circ H \xrightarrow{\gamma \circ 1} (\Sigma G_1) \circ H \longrightarrow \dots$$

where  $\Sigma(G_1 \circ H) \simeq G_1 \wedge H \simeq (\Sigma G_1) \circ H$ . Since this sequence also induces an exact sequence in homology it is a cofibration sequence. The other case is similar.  $\square$

**Proposition 5.7.** *If  $f: G \rightarrow H$  is an index  $p$  approximation, the maps  $f \circ 1: G \circ L \rightarrow H \circ L$  and  $1 \circ f: L \circ G \rightarrow L \circ H$  are index  $p$  approximations as well.*

*Proof.* Factor  $g$  as

$$\Sigma H \xrightarrow{g'} \Sigma^2 K \xrightarrow{[p]} \Sigma^2 K$$

and consider the diagram

$$\begin{array}{ccc} (\Sigma H) \circ L & \xrightarrow{g \circ 1} & (\Sigma^2 K) \circ L \simeq \Sigma^2(K \circ L) \\ = \downarrow & & \uparrow [p] \circ 1 \quad \uparrow [p] \\ (\Sigma H) \circ L & \xrightarrow{g' \circ 1} & (\Sigma^2 K) \circ L \simeq \Sigma^2(K \circ L) \end{array}$$

where the righthand square commutes by 5.5. The map

$$g \circ 1: (\Sigma H) \circ L \rightarrow (\Sigma^2 K) \circ L$$

is the cofiber of  $f \circ 1$ , and  $g' \circ 1$  is a co- $H$  map since  $g'$  is a co- $H$  map. Thus  $f \circ 1$  is an index  $p$  approximation. The other case is similar.  $\square$

**Corollary 5.8.** *Suppose  $G \xrightarrow{f} H$  is an index  $p$  approximation. Then*

$$f^{[i]}: G^{[i]} \rightarrow H^{[i]}$$

*is an iterated index  $p$  approximation.*

*Proof.* We first observe that

$$G \circ H^{[j]} \xrightarrow{f \circ 1} H \circ H^{[j]} = H^{[j+1]}$$

is an index  $p$  approximation by 5.7. We then see by induction that

$$G^{[i]} H^{[j]} = G \circ (G^{[i-1]} H^{[j]}) \rightarrow G \circ (G^{[i-2]} H^{[j+1]}) = G^{[i-1]} H^{[j+1]}$$

is an index  $p$  approximation. Finally

$$G^{[i]} \rightarrow G^{[i-1]} H$$

is an iterated index  $p$  approximation by induction since it factors as

$$G^{[i]} = G \circ (G^{[i-1]}) \rightarrow G \circ (G^{[i-2]} H) = G^{[i-1]} H.$$

Consequently

$$G^{[i]} \rightarrow G^{[i-1]}H \rightarrow G^{[i-2]}H^{[2]} \rightarrow \dots \rightarrow G \circ H^{[i-1]} \rightarrow H^{[i]}$$

is an iterated index  $p$  approximation.  $\square$

**Theorem 5.9.** *Suppose*

$$\Omega X \rightarrow E \rightarrow G$$

*is a principal fibration classified by a map  $\varphi: G \rightarrow X$  where  $X$  is a homotopy commutative  $H$ -space. Suppose  $f: H \rightarrow G$  is an index  $p$  approximation. Then, for any map  $\nu: E \rightarrow BW_n$  the compositions*

$$\Sigma(\Omega G \wedge \Omega G) \xrightarrow{\Gamma} E \xrightarrow{\nu} BW_n$$

*is null homotopic iff the composition*

$$\Sigma(\Omega H \wedge \Omega H) \xrightarrow{\Sigma(\Omega f \wedge \Omega f)} \Sigma(\Omega G \wedge \Omega G) \xrightarrow{\Gamma} E \xrightarrow{\nu} BW_n$$

*is null homotopic.*

*Proof.* Suppose the composition

$$\Sigma(\Omega H \wedge \Omega H) \rightarrow \Sigma(\Omega G \wedge \Omega G) \xrightarrow{\Gamma} E \xrightarrow{\nu} BW_n$$

is null homotopic. Since  $f: H \rightarrow G$  is a co- $H$  map, there is a homotopy commutative diagram:

$$\begin{array}{ccc} H^{[i]} & \xrightarrow{\widetilde{ad}_\times^i} & \Sigma(\Omega H \wedge \Omega H) \\ f^{[i]} \downarrow & & \downarrow \\ G^{[i]} & \xrightarrow{\widetilde{ad}_\times^i} & \Sigma(\Omega G \wedge \Omega G). \end{array}$$

Since  $f$  is an index  $p$  approximation,  $f^{[i]}$  is an iterated index  $p$  approximation by 5.8; thus the compositions

$$G^{[i]} \xrightarrow{\widetilde{ad}_\times^i} \Sigma(\Omega G \wedge \Omega G) \xrightarrow{\Gamma} E \xrightarrow{\nu} BW_n$$

are null homotopic for all  $i \geq 1$ . The result then follows from 3.8.  $\square$

We will use this result to transfer conditions on  $\nu_k$  to the composition:

$$\Sigma(\Omega L_k \wedge \Omega L_k) \longrightarrow \Sigma(\Omega G_k \wedge \Omega G_k) \xrightarrow{\Gamma_k} E_k.$$

We need to iterate this. We have to consider the issue that for  $\zeta_k: L_k \rightarrow G_k$  to be a co- $H$  map, we need to use an exotic co- $H$  space structure on  $L_k$ . We will show that the triviality of the composition above does not depend on the co- $H$  space structure of  $L_k$ . To see this, recall that the map  $\Gamma_k: \Sigma(\Omega G_k \wedge G_k) \rightarrow E_k$  was defined in section 2 based on the fact that  $E_k$  was defined by a principal fibration

$$\Omega S^{2n+1}\{p^r\} \rightarrow E_k \rightarrow G_k$$

classified by a map  $\varphi_k: G_k \rightarrow S^{2n+1}\{p^r\}$  where  $S^{2n+1}\{p^r\}$  is a homotopy commutative  $H$ -space with  $H$ -space structure map chosen to have a strict unit and (in 2.6) a commuting homotopy which is constant on the axes. The fact that  $G_k$  is a co- $H$  space was not used.

For any space  $X$  and map  $\zeta: X \rightarrow G_k$ , we can construct the pullback

$$\begin{array}{ccc} \Omega S^{2n+1}\{p^r\} & \xlongequal{\quad} & \Omega S^{2n+1}\{p^r\} \\ \downarrow & & \downarrow \\ E(X) & \xrightarrow{\quad} & E_k \\ \downarrow & & \downarrow \\ X & \xrightarrow{\quad \zeta \quad} & G_k \end{array}$$

which is induced by the composition  $\varphi_k \zeta$ . Consequently there is a map  $\Gamma(X): \Sigma(\Omega X \wedge \Omega X) \rightarrow P$  and a strictly commutative diagram:

$$\begin{array}{ccc} \Sigma(\Omega X \wedge \Omega X) & \longrightarrow & \Sigma(\Omega G_k \wedge \Omega G_k) \\ \Gamma(X) \downarrow & & \Gamma_k \downarrow \\ E(X) & \longrightarrow & E_k. \end{array}$$

Consider the homotopy equivalence

$$X_k = G_{k-1} \vee P^{2np^k}(p^{r+k-1}) \vee P^{2np^k+1}(p^{r+k-1}) \rightarrow L_k$$

where we give  $X_k$  the split co- $H$  space structure, so this map is not a co- $H$  map. Nevertheless, we have a strictly commutative diagram

$$\begin{array}{ccccc} \Sigma(\Omega X_k \wedge \Omega X_k) & \xrightarrow{\cong} & \Sigma(\Omega L_k \wedge \Omega L_k) & \longrightarrow & \Sigma(\Omega G_k \wedge \Omega G_k) \\ \Gamma(X_k) \downarrow & & \Gamma(L_k) \downarrow & & \Gamma_k \downarrow \\ E(X_k) & \xrightarrow{\cong} & E(L_k) & \longrightarrow & E_k \\ \downarrow & & \downarrow & & \downarrow \pi_k \\ X_k & \xrightarrow{\cong} & L_k & \xrightarrow{\zeta_k} & G_k \end{array}$$

since  $\zeta_k$  is an iterated index  $p$  approximation, we have

**Proposition 5.10.** *For any map  $\nu: E_k \rightarrow BW_n$ ,  $\nu\Gamma_k$  is null homotopic iff the composition*

$$\Sigma(\Omega X_k \wedge \Omega X_k) \xrightarrow{\Gamma(X_k)} E(X_k) \simeq E(L_k) \rightarrow E_k \xrightarrow{\nu} BW_n$$

*is null homotopic.*

□

We now define spaces with split co- $H$  space structures (coproducts in the category of co- $H$  spaces):

$$\begin{aligned} W(j, k) &= \bigvee_{i=j}^k P^{2np^i}(p^{r+i-1}) \vee P^{2np^i+1}(p^{r+i-1}) \\ G(j, k) &= G_j \vee W(j+1, k) \\ L(j, k) &= L_j \vee W(j+1, k). \end{aligned}$$

Consequently we have homotopy equivalences

$$G(j, k) \simeq L(j+1, k)$$

which are not co- $H$  equivalences, and index  $p$  approximations

$$L(j, k) \xrightarrow{\zeta_j \vee 1} G(j, k).$$

This leads to a chain:

$$f: G(0, k) \simeq L(1, k) \rightarrow G(1, k) \simeq L(2, k) \rightarrow \cdots \rightarrow G(k-1, k) \simeq L_k \rightarrow G_k.$$

**Theorem 5.11.** *For any given map  $\nu: E_k \rightarrow BW_n$  the composition*

$$\Sigma(\Omega G_k \wedge \Omega G_k) \xrightarrow{\Gamma_k} E_k \xrightarrow{\nu} BW_n$$

*is null homotopic iff the composition*

$$\Sigma(\Omega G(0, k) \wedge \Omega G(0, k)) \xrightarrow{\Sigma(\Omega f \wedge \Omega f)} \Sigma(\Omega G_k \wedge \Omega G_k) \xrightarrow{\Gamma_k} E_k \xrightarrow{\nu} BW_n$$

*is null homotopic, where*

$$G(0, k) = P^{2n+1} \vee \bigvee_{i=1}^k P^{2np^i}(p^{r+i-1}) \vee P^{2np^i+1}(p^{r+i-1})$$

*and the map  $f: G(0, k) \rightarrow G_k$  is defined by the inclusion of*

$$P^{2n+1} = G_0 \rightarrow G_k$$

*and the maps  $\pi_k c(i)$  and  $\pi_k a(i)$  for  $1 \leq i \leq k$ :*

$$\begin{aligned} P^{2np^i}(p^{r+i-1}) &\xrightarrow{a(i)} E_i \rightarrow E_k \xrightarrow{\pi_k} G_k \\ P^{2np^i+1}(p^{r+i-1}) &\xrightarrow{c(i)} E_i \rightarrow E_k \xrightarrow{\pi_k} G_k. \end{aligned}$$

□

Observe the diagram

$$\begin{array}{ccccc} & & E(0, k) & \longrightarrow & E_k \\ & \nearrow \Gamma & \downarrow & & \downarrow \\ \Sigma(\Omega G(0, k) \wedge \Omega G(0, k)) & \longrightarrow & G(0, k) & \xrightarrow{f} & G_k \end{array}$$

where  $\Gamma$  is  $\Gamma_k$  composed with  $\Sigma(\Omega f \wedge \Omega f)$ . Since  $G(0, k)$  is a wedge of Moore spaces, the components of  $\Gamma$  are  $H$ -space based Samelson products as defined by Neisendorfer [Nei10a]. This will be studied in the next section.



## 6. REDUCTION

Recall that the inductive assumption is that we have constructed a retraction  $\nu_{k-1}: E_{k-1} \rightarrow BW_n$  such that  $\nu_{k-1}\Gamma_{k-1}$  is null homotopic. In section 4 we saw that we can construct classes  $a(i)$ ,  $c(i)$ , and  $\beta_i$  for  $i \leq k$  and in section 5 we reduced the constraints on the construction of  $\nu_k$  to a condition involving the maps  $a(i)$  and  $c(i)$ .

We now make a further simplification by burying the classes  $c(i)$  in the base space. Specifically, we define a map

$$c: C_k = \bigvee_{i=1}^k P^{2np^i+1}(p^{r+i-1}) \rightarrow E_k$$

by the compositions

$$P^{2np^i+1}(p^{r+i-1}) \xrightarrow{c(i)} E_i \longrightarrow E_k,$$

and define  $D_k$  by a cofibration

$$C_k \xrightarrow{\pi_k c} G_k \longrightarrow D_k.$$

**Proposition 6.1.** *There is a homotopy commutative diagram of cofibration sequences*

$$\begin{array}{ccccc} P^{2np^k+1}(p^{r+k-1}) & \xrightarrow{\rho} & P^{2np^k+1}(p^{r+k}) & \xrightarrow{\sigma^{r+k-1}} & P^{2np^k+1}(p) \\ \uparrow & & \uparrow \pi & & \uparrow \\ C_k & \xrightarrow{\pi_k c} & G_k & \longrightarrow & D_k \\ \uparrow & & \uparrow & & \uparrow \\ C_{k-1} & \xrightarrow{\pi_{k-1} c} & G_{k-1} & \longrightarrow & D_{k-1} \end{array}$$

and

$$H_i(D_k) = \begin{cases} Z/p^r & \text{if } i = 2n \\ Z/p & \text{if } i = 2np^j, 1 \leq j \leq k \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* The composition

$$P^{2np^k+1}(p^{r+k-1}) \xrightarrow{c(k)} E_k \xrightarrow{\pi_k} G_k \xrightarrow{\pi} P^{2np^k+1}(p^{r+k})$$

is  $\rho$  by 4.4. The homology calculation is immediate.  $\square$

Since  $\varphi_k \pi_k$  is null homotopic, we can extend  $\varphi_k$  to a map

$$\varphi'_k: D_k \rightarrow S^{2n+1}\{p^r\}.$$

Any such extension defines a diagram of vertical fibration sequences:

$$(6.2) \quad \begin{array}{ccccc} E_k & \xrightarrow{\tau_k} & J_k & \xrightarrow{\eta_k} & F_k \\ \pi_k \downarrow & & \xi_k \downarrow & & \sigma_k \downarrow \\ G_k & \longrightarrow & D_k & \xlongequal{\quad} & D_k \\ \varphi_k \downarrow & & \varphi'_k \downarrow & & \downarrow \\ S^{2n+1}\{p^r\} & \longrightarrow & S^{2n+1}\{p^r\} & \longrightarrow & S^{2n+1} \end{array}$$

**Proposition 6.3.** *We can choose an extension  $\varphi'_k$  of  $\varphi'_{k-1}$  in such a way that the composition*

$$P^{2np^k+1}(p^{r+k-1}) \xrightarrow{c(k)} E_k \xrightarrow{\tau_k} J_k$$

*is null homotopic.*

*Proof.* We begin by defining  $D$  by a pushout square:

$$\begin{array}{ccc} G_k & \longrightarrow & D \\ \uparrow & & \uparrow \\ G_{k-1} & \longrightarrow & D_{k-1} \end{array}$$

Using the lower righthand square in 6.1 and the pushout property, we see that there is a cofibration

$$P^{2np^k+1}(p^{r+k-1}) \xrightarrow{\alpha} D \longrightarrow D_k$$

where  $\alpha$  is the composition:

$$P^{2np^k+1}(p^{r+k-1}) \xrightarrow{c(k)} E_k \xrightarrow{\pi_k} G_k \longrightarrow D.$$

We use the pushout property to construct  $\varphi: D \rightarrow S^{2n+1}\{p^r\}$  by  $\varphi'_{k-1}$  on  $D_{k-1}$  and  $\varphi_k$  on  $G_k$ . We seek a map  $\varphi'_k$  in the diagram

$$\begin{array}{ccccccc} E_k & \longrightarrow & J & \longrightarrow & J_k & & \\ \pi_k \downarrow & & \downarrow & & \downarrow & & \\ G_k & \longrightarrow & D & \longrightarrow & D_k = D \cup_{\alpha} CP^{2np^k+1}(p^{r+k-1}) & & \\ \varphi_k \downarrow & & \varphi \downarrow & & \varphi'_k \downarrow & & \\ S^{2n+1}\{p^r\} & \xlongequal{\quad} & S^{2n+1}\{p^r\} & \xlongequal{\quad} & S^{2n+1}\{p^r\} & & \end{array}$$

We assert that we can choose  $\varphi'_k$  so that the composition

$$P^{2np^k+1}(p^{r+k-1}) \xrightarrow{c(k)} E_k \longrightarrow J \longrightarrow J_k$$

is null homotopic. Note that  $\alpha$  is homotopic to the composition

$$P^{2np^k+1}(p^{r+k-1}) \xrightarrow{c(k)} E_k \longrightarrow J \longrightarrow D.$$

The assertion then follows from

**Lemma 6.4.** *Suppose  $J \xrightarrow{\pi} D$  is a principal fibration induced by a map  $\varphi: D \rightarrow S$ . Suppose  $c: Q \rightarrow J$  and  $D'$  is the mapping cone of  $\pi c$ . Then there is a map  $\varphi': D' \rightarrow S$  with homotopy fiber  $J'$  such that the composition  $Q \longrightarrow J \longrightarrow J'$  is null homotopic in the diagram:*

$$\begin{array}{ccccc} Q & \xrightarrow{c} & J & \longrightarrow & J' \\ & & \downarrow \pi & & \downarrow \\ & & D & \longrightarrow & D' \\ & & \downarrow \varphi & & \downarrow \varphi' \\ & & S & \xlongequal{\quad} & S. \end{array}$$

*Proof.*  $J = \{(d, \omega) \in D \times S^I \mid w(0) = x \text{ and } \omega(1) = \varphi(d)\}$  so  $c(q)$  has components  $(c_1(q), c_2(q))$  where  $c_1(q) \in D$ ,  $c_2(q) \in S^I$  with  $c_2(q)(0) = *$  and  $c_2(q)(1) = \varphi(c_1(q))$ . Write  $D' = D \cup_{\alpha} CQ$  with 0 at the vertex of the cone and  $\alpha(q) = c_1(q)$ . Now define  $\varphi': D' \rightarrow S$  by  $\varphi'(d) = \varphi(d)$  for  $d \in D$  and  $\varphi'(q, t) = c_2(q)(t)$ . This is well defined and we can define a homotopy

$$H: Q \times I \rightarrow J' \subset D' \times S^I$$

by the formula

$$H(q, t) = ((q, t), c_2(q)_t)$$

where  $c_2(q)_t$  is the path defined as  $c_2(q)_t(s) = c_1(q)(st)$ . □

This proves the lemma and hence the proposition. □

Now define

$$(6.5) \quad U_k = P^{2n+1} \vee \bigvee_{i=1}^k P^{2np^i}(p^{r+i-1}),$$

so  $G(0, k) = U_k \vee C_k$ , and we have a homotopy commutative square

$$\begin{array}{ccc} G(0, k) & \longrightarrow & G_k \\ \downarrow & & \downarrow \\ U_k & \xrightarrow{a} & D_k \end{array}$$

where the lefthand vertical map is the projection and  $a$  is defined on  $P^{2np^i}(p^{r+i-1})$  as the composition:

$$p^{2np^i}(p^{r+i-1}) \xrightarrow{a(i)} E_k \xrightarrow{\tau_k} J_k \xrightarrow{\xi_k} D_k.$$

From this we construct homotopy commutative diagram:

$$(6.6) \quad \begin{array}{ccccc} \Sigma(\Omega G(0, k) \wedge \Omega G(0, k)) & \longrightarrow & \Sigma(\Omega G_k \wedge \Omega G_k) & \xrightarrow{\Gamma_k} & E_k \\ \downarrow & & \downarrow & & \downarrow \tau_k \\ \Sigma(\Omega U_k \wedge \Omega U_k) & \longrightarrow & \Sigma(\Omega D_k \wedge \Omega D_k) & \xrightarrow{\Gamma} & J_k \end{array}$$

**Proposition 6.7.** *Suppose there is a retraction  $\gamma_k: J_k \rightarrow BW_n$  such that the compositions*

$$U_k^{[j]} \xrightarrow{\widetilde{ad}_\times^j(a)} J_k \xrightarrow{\gamma_k} BW_n$$

*are null homotopic for each  $j \geq 2$ . Then the composition*

$$\Sigma(\Omega G_k \wedge \Omega G_k) \xrightarrow{\Gamma_k} E_k \xrightarrow{\tau_k} J_k \xrightarrow{\gamma_k} BW_n$$

*is null homotopic and the induction is complete.*

*Proof.* By (6.6) and 5.11, it suffices to show that the composition

$$\Sigma(\Omega U_k \wedge \Omega U_k) \longrightarrow \Sigma(\Omega D_k \wedge \Omega D_k) \xrightarrow{\Gamma} J_k \xrightarrow{\gamma_k} BW_n$$

is null homotopic. Define  $E(U_k)_k$  as a pullback:

$$\begin{array}{ccc} E(U_k) & \longrightarrow & J_k \\ \downarrow & & \downarrow \\ U_k & \xrightarrow{a} & D_k \end{array}$$

Then by naturality, it suffices to show that the composition

$$\Sigma(\Omega U_k \wedge \Omega U_k) \xrightarrow{\Gamma(U_k)} E(U_k) \longrightarrow J_k \xrightarrow{\gamma_k} BW_n$$

is null homotopic. But since  $U_k$  is a co- $H$  space, we can apply 3.8 to finish the proof.  $\square$

Now write  $U_k = \Sigma P_k$  where

$$P_k = P^{2n} \vee \bigvee_{i=1}^k P^{2np^i-1}(p^{r+i-1})$$

so  $U_k^{[j]} = \Sigma P_k \wedge \cdots \wedge P_k$  by 3.2. According to 3.7, the map

$$\Sigma P_k^{(j)} \xrightarrow{\widetilde{ad}_\times^j(a)} J_k$$

is the iterated external  $H$ -space based Whitehead product  $\{a, \dots, a\}_\times$ , where  $\{x_1, \dots, x_j\}_\times$  is defined inductively by the formula:

$$\begin{aligned} \{x_1, \dots, x_j\}_\times &= \{x_1, \xi_k \{x_2, \dots, x_j\}_\times\}_\times \\ &= \{x_1, \{x_2, \dots, x_k\}\}_\times. \end{aligned}$$

If  $j > 2$  this is equal to the relative Whitehead product  $\{x_1, \{x_2, \dots, x_j\}_\times\}_r$  by 3.10.

Since the smash product of Moore spaces is a wedge of Moore spaces, we have

**Proposition 6.8.** *The map*

$$\widetilde{ad}_\times^j(a): U_k^{[j]} = \Sigma P_k^{(j)} \rightarrow J_k$$

*when restricted to one of the  $(k+1)^j$  iterated smash products of Moore spaces is an iterated external  $H$ -space based Whitehead product*

$$\{x_1, \dots, x_j\}_\times$$

*where each  $x_i$  is either  $\xi_k a(i): P^{2np^i-1}(p^{r+i-1}) \rightarrow D_k$  for  $1 \leq i \leq k$  or the inclusion  $P^{2n+1} \rightarrow D_k$ .  $\square$*

In this analysis, we are considering smash products of Moore spaces for distinct cyclic groups. It is necessary, therefore, to split these smash products and resolve the external Whitehead products into internal Whitehead products on the factors.

Choose a map

$$\Delta: P^{m+n}(p^s) \rightarrow P^m(p^s) \wedge P^n(p^s)$$

so that the diagram

$$\begin{array}{ccc} P^{m+n}(p^s) & \xrightarrow{\Delta} & P^m(p^s) \wedge P^n(p^s) \\ \pi_{m+n} \downarrow & & \downarrow \\ S^{m+n} & \xrightarrow{\simeq} & S^m \wedge S^n \end{array}$$

commutes up to homotopy. Such a choice is possible when  $m, n \geq 2$  for  $p$  odd and is unique up to homotopy.

Neisendorfer [Nei80] has produced internal Whitehead and Samelson products for homotopy with  $Z/p^s$  coefficients. The Whitehead product of  $x \in \pi_{m+1}(X; Z/p^s)$  and  $y \in \pi_{n+1}(X; Z/p^s)$  is an element  $[x, y] \in \pi_{m+n+1}(X; Z/p^s)$  defined as the homotopy class of the composition:

$$\begin{aligned} (6.9) \quad P^{m+n+1}(p^s) &= \Sigma P^{m+n}(p^s) \xrightarrow{\Sigma \Delta} \Sigma P^m(p^s) \wedge P^n(p^s) \\ &= P^{m+1}(p^s) \circ P^{n+1}(p^s) \xrightarrow{\{x, y\}} X \end{aligned}$$

As we will need to consider such pairings with different coefficients, suppose  $x \in \pi_{m+1}(X; Z/p^r)$  and  $y \in \pi_{n+1}(X; Z/p^{r+t})$ . We can still form the external Whitehead product:

$$\Sigma P^m(p^r) \wedge P^n(p^{r+t}) = P^{m+1}(p^r) \circ P^{n+1}(p^{r+t}) \xrightarrow{\{x, y\}} X.$$

Since the map of degree  $p^{r+t}$  on  $P^m(p^r)$  is null homotopic, there is a splitting:

$$P^m(p^r) \wedge P^n(p^{r+t}) \simeq P^{m+n}(p^r) \vee P^{m+n+1}(p^r).$$

We now choose an explicit splitting. Recall  $\delta_t = \beta\rho^t$ .

**Proposition 6.10.** *There is a splitting of  $P^m(p^r) \wedge P^n(p^{r+t})$  defined by the two compositions:*

$$\begin{aligned} P^{m+n}(p^r) &\xrightarrow{\Delta} P^m(p^r) \wedge P^n(p^r) \xrightarrow{1 \wedge \rho^t} P^m(p^r) \wedge P^n(p^{r+t}) \\ P^{m+n-1}(p^r) &\xrightarrow{\Delta} P^m(p^r) \wedge P^{n-1}(p^r) \xrightarrow{1 \wedge \delta_t} P^m(p^r) \wedge P^n(p^{r+t}) \end{aligned}$$

*Proof.*  $(1 \wedge \pi_n)(1 \wedge \rho^t)\Delta = (1 \wedge \pi_n)\Delta$  induces a mod  $p$  homology isomorphism, so  $(1 \wedge \rho^t)\Delta$  induces a homology monomorphism. The second composition factors

$$\begin{aligned} P^{m+n-1}(p^r) &\xrightarrow{\Delta} P^m(p^r) \wedge P^{n-1}(p^r) \xrightarrow{1 \wedge \pi_{n-1}} P^m(p^r) \wedge S^{n-1} \\ &\xrightarrow{1 \wedge \iota_{n-1}} P^m(p^r) \wedge P^{n-1}(p^{r+t}) \end{aligned}$$

and the composition of the first two maps is a homotopy equivalence. Since the third map induces a mod  $p$  homology monomorphism, this composition does as well. Counting ranks, we see that the two maps together define a homotopy equivalence:

$$e: P^{m+n}(p^r) \vee P^{m+n-1}(p^r) \xrightarrow{\simeq} P^m(p^r) \wedge P^n(p^{r+t}) \quad \square$$

We apply this to the internal Whitehead product (6.9) to get

**Proposition 6.11.**

$$\{x, y\}e = [x, y\rho^t] \vee [x, y\delta_t]: P^{m+n}(p^r) \vee P^{m+n-1}(p^r) \rightarrow X. \quad \square$$

Suppose now that we are given a principal fibration

$$\Omega X \rightarrow E \rightarrow B$$

classified by a map  $\varphi: B \rightarrow X$  where  $X$  is a homotopy commutative  $H$ -space with strict unit and we are given classes  $u \in \pi_m(B; Z/p^r)$  and  $v \in \pi_n(B; Z/p^{r+t})$ . Then we have

**Proposition 6.12.**

$$\{u, v\}_\times e = [u, \rho^t v]_\times \vee [u, v\delta_t]_\times: P^{m+n}(p^r) \vee P^{m+n-1}(p^{r+t}) \rightarrow E.$$

*Proof.* Both  $\{u, v\}_\times e$  and  $[u, v\rho^t]_\times \vee [u, v\delta_t]_\times$  represent maps:

$$P^{m+n}(p^r) \vee P^{m+n-1}(p^r) \rightarrow \Sigma(\Omega B \wedge \Omega B)$$

which are homotopic after projection

$$\Sigma(\Omega B \wedge \Omega B) \xrightarrow{\omega} B \vee B$$

by 6.11. Since  $\Omega\omega$  has a left homotopy universe, these maps are homotopic. Composing with  $\Gamma: \Sigma(\Omega B \wedge \Omega B) \rightarrow E$  finishes the proof.  $\square$

**Theorem 6.13.** *The restriction of the map*

$$\widetilde{ad}_\times^j(a): \Sigma P_k \wedge \cdots \wedge P_k \rightarrow J_k$$

*to any Moore space in any decomposition of  $\Sigma P_k \wedge \cdots \wedge P_k$  is homotopic to a linear combination of weight  $j$  iterated internal  $H$ -space based Whitehead products*

$$[x_1, \dots, x_j]_\times$$

*where each  $x_i$  is one of the following:  $\xi_k \tau_k a(i) \rho^t$ ,  $\xi_k \tau_k a(i) \delta_t$ ,  $\mu$ ,  $\nu$  for  $1 \leq i \leq k$  and for appropriate values of  $t$ .*

*Proof.* This is an easy induction on  $j$ . □

Similar to 6.12, we have

**Proposition 6.14.**  $\{x, u\}_r e = [x, u \rho^t]_r \vee [x, u \delta_t]_r$  where  $x \in \pi_m(B; Z/p^r)$  and  $u \in \pi_n(E; Z/p^{r+t})$ . □

There is one special case of this that we will need in section 10. This involves relative Whitehead products  $[x, u]_r$  when  $u: S^n \rightarrow E$  and  $x: P^m \rightarrow B$ . In this case

$$[x, u]_r = \{x, u\}_r: P^m \circ S^n \rightarrow E.$$

**Proposition 6.15.**  $[x, u \pi_n]_r = [x, u]_r: P^{m+n-1} \rightarrow E$ .

*Proof.*  $\{x, u \pi_n\}_r e = [x, u \pi_n]_r \vee 0$  since  $\pi_n \delta_t = 0$ . Consequently we have a homotopy commutative diagram

$$\begin{array}{ccccc} P^{m+n-1} \vee P^{m+n-2} & \xrightarrow{e} & P^m \circ P^n & \xrightarrow{1 \circ \pi_n} & P^m \circ S^n \simeq P^{m+n-1} \\ & \searrow [x, u \pi_n]_r \vee 0 & \downarrow \{x, u \pi_n\}_r & \swarrow [x, u]_r & \\ & & E & & \end{array}$$

where the upper composition is homotopic to projection onto the first factor. □

## 7. CONGRUENCE HOMOTOPY THEORY

The results of section 6, and in particular 6.13, indicate that the obstructions to constructing a suitable retraction  $\nu_k = \gamma_k \tau_k$  are mod  $p^{r+t}$  homotopy classes in  $J_k$ . These obstructions are iterated  $H$ -space based Whitehead products in  $J_k$ . Since  $BW_n$  is an  $H$ -space, any Whitehead products of classes in  $J_k$  will be annihilated by any such map  $\gamma_k$ . We are led to a coarser classification.

**Definition 7.1.** Two maps  $f, g: X \rightarrow Y$  will be called congruent (written  $f \equiv g$ ) if  $\Sigma f$  and  $\Sigma g$  are homotopic in  $[\Sigma X, \Sigma Y]$ . We write  $e[X, Y]$  for the set of congruence classes of pointed maps:  $X \rightarrow Y$  and

$$e\pi_k(Y; Z/p^s) = e[P^k(p^s), Y].$$

Clearly congruence is an equivalence relation and composition is well defined on congruence classes. This defines the congruence homotopy category. It is easy to prove

**Proposition 7.2.** *Suppose  $f \equiv g: X \rightarrow Y$  and  $h: Y \rightarrow Z$  where  $Z$  is an  $H$ -space.  $hf$  and  $hg$  are homotopic.*

Consequently, it is sufficient to classify the iterated  $H$ -space based Whitehead products of 6.13 up to congruence. We will also need to consider congruence in a different way in section 10. In constructing  $\gamma_k$  we will make alterations in dimensions where obstructions of level  $k - 1$  may resurface. This is a delicate point which has needed much attention. For this reason we need to develop deeper properties of congruence homotopy theory.

**Proposition 7.3.** *The inclusion  $Y_1 \vee Y_2 \xrightarrow{i} Y_1 \times Y_2$  defines a 1-1 map  $\iota_*: e[X, Y_1 \vee Y_2] \rightarrow e[X, Y_1 \times Y_2]$ . Furthermore, if  $G$  is a co- $H$  space  $e[G, X]$  is an Abelian group and*

$$e[G, Y_1 \vee Y_2] \cong e[G, Y_1 \times Y_2] \cong e[G, Y_1] \oplus e[G, Y_2].$$

*Proof.* Suppose  $f, g: X \rightarrow Y_1 \vee Y_2$  and the compositions:

$$\begin{aligned} \Sigma X &\xrightarrow{\Sigma f} \Sigma(Y_1 \vee Y_2) \xrightarrow{\Sigma i} \Sigma(Y_1 \times Y_2) \\ \Sigma X &\xrightarrow{\Sigma g} \Sigma(Y_1 \vee Y_2) \xrightarrow{\Sigma i} \Sigma(Y_1 \times Y_2) \end{aligned}$$

are homotopic. Since  $\Sigma i$  has a left homotopy inverse,  $\Sigma f$  and  $\Sigma g$  are homotopic. The co- $H$  space structure on  $G$  defines a multiplication on  $[G, X]$  and the map

$$[G, X] \rightarrow [G, \Omega \Sigma X]$$

is multiplicative. However  $[G, \Omega \Sigma X]$  is an Abelian group by a standard argument. Since  $e[G, X]$  is a subgroup of  $[G, \Omega \Sigma X] \cong [\Sigma G, \Sigma X]$ , it also is Abelian. Finally observe that the composition

$$e[G, Y_1] \oplus e[G, Y_2] \rightarrow e[G, Y_1 \vee Y_2] \rightarrow e[G, Y_1 \times Y_2] \rightarrow e[G, Y_1] \oplus e[G, Y_2]$$

is the identity where the first and last maps are defined by naturality. Thus the composition of the first two is 1-1. But this composition is also onto since any element of  $e[G, Y_1 \times Y_2]$  is represented by a map  $G \rightarrow Y_1 \times Y_2$  so all these maps are isomorphisms.  $\square$

**Proposition 7.4.** *Suppose  $G$  and  $H$  are co- $H$  spaces. Then composition defines a homomorphism:*

$$e[G, H] \otimes e[H, X] \rightarrow e[G, X].$$

*Proof.* The only issue is the distributive law

$$(\beta_1 + \beta_2)\alpha \equiv \beta_1\alpha + \beta_2\alpha$$



for  $\alpha: G \rightarrow H$  and  $\beta_1, \beta_2: H \rightarrow X$ . To prove this we show that the following diagram commutes up to congruence:

$$\begin{array}{ccc} G & \xrightarrow{\alpha} & H \\ \downarrow & \equiv & \downarrow \\ G \vee G & \xrightarrow{\alpha \vee \alpha} & H \vee H. \end{array}$$

This certainly commutes after the inclusion of  $H \vee H \rightarrow H \times H$ . Thus it commutes up to congruence by 7.3.  $\square$

This will be useful when  $G$  and  $H$  are Moore spaces.

**Corollary 7.5.** *The category of co- $H$  spaces and congruence classes of continuous maps is an additive category.*

*Proof.*  $G \vee H$  is both a product and co-product by 7.3 and composition is bilinear by 7.4.  $\square$

**Theorem 7.6.** *Suppose  $\varphi: \Sigma^2 X \rightarrow P^{2m}(p^{r+t})$  has order  $p^r$ . Then there are maps  $\varphi_1: \Sigma^2 X \rightarrow P^{2m}(p^r)$  and  $\varphi_2: \Sigma^2 X \rightarrow S^{2m-1}$  such that*

$$\varphi \equiv \varphi_1 \rho^t + \iota_{2m-1} \varphi_2.$$

*Proof.* According to [CMN79c, 11.1] or [Gra99, 1.2], there is a fibration sequence

$$\Omega P^{2m}(p^{r+t}) \xrightarrow{\partial} S^{2m-1}\{p^{r+t}\} \longrightarrow W \xrightarrow{\pi} P^{2m}(p^{r+t})$$

where  $W$  is a  $4m - 3$  connected wedge of Moore spaces and  $\pi$  is an iterated Whitehead product on each factor. In particular,  $\pi$  is null congruent. A right homotopy inverse for  $\partial$  is constructed as follows. Given any map  $\theta: U \rightarrow V$ , there is a natural map from the fiber of  $\theta$  to the loop space on the cofiber:

$$\Phi: F_\theta \rightarrow \Omega(V \cup_\theta CU).$$

This defines a map  $\Phi: S^{2m-1}\{p^{r+t}\} \rightarrow \Omega P^{2m}(p^{r+t})$  and  $\partial\Phi$  is a homology equivalence since  $S^{2m-1}\{p^{r+t}\}$  is atomic. This defines a splitting of  $\Omega P^{2m}(p^{r+t})$  and we have a direct sum decomposition

$$[\Sigma^2 X, W] \oplus [\Sigma X, S^{2m-1}\{p^{r+t}\}] \longrightarrow [\Sigma^2 X, P^{2m}(p^{r+t})]$$

$$(\alpha, \beta) \longleftrightarrow \varphi = \pi\alpha + \widetilde{\Phi}\beta$$

where  $\widetilde{\Phi}\beta$  is the adjoint of  $\Phi\beta: \Sigma X \rightarrow \Omega P^{2m}(p^{r+t})$ . Since  $\varphi$  has order  $p^r$ , both  $\alpha$  and  $\beta$  have order  $p^r$ . Since  $\pi$  is null congruent, we have

$$\varphi \equiv \widetilde{\Phi}\beta.$$

Now consider the diagram of fibration sequences:

$$\begin{array}{ccccc}
 F & \xrightarrow{f} & S^{2m-1}\{p^{r+t}\} & \xrightarrow{p^r} & S^{2m-1}\{p^{r+t}\} \\
 \downarrow & & \downarrow & & \downarrow \\
 S^{2m-1}\{p^r\} & \longrightarrow & S^{2m-1} & \xrightarrow{p^r} & S^{2m-1} \\
 \downarrow * \sim p^{r+t} & & \downarrow p^{r+t} & & \downarrow p^{r+t} \\
 S^{2m-1}\{p^r\} & \longrightarrow & S^{2m-1} & \xrightarrow{p^r} & S^{2m-1}.
 \end{array}$$

From this we see that  $F \simeq S^{2m-1}\{p^r\} \times \Omega S^{2m-1}\{p^r\}$ . We choose a splitting of  $F$  as follows: define a map  $\rho^t$  by the diagram of vertical fibration sequences:

$$\begin{array}{ccc}
 S^{2m-1}\{p^r\} & \xrightarrow{\rho^t} & S^{2m-1}\{p^{r+t}\} \\
 \downarrow & & \downarrow \\
 S^{2m-1} & \xlongequal{\quad} & S^{2m-1} \\
 \downarrow p^r & & \downarrow p^{r+t} \\
 S^{2m-1} & \xrightarrow{p^t} & S^{2m-1}.
 \end{array}$$

The map  $\rho^t$  factors through  $f$  and defines a splitting. Thus the composition

$$S^{2m-1}\{p^r\} \times \Omega S^{2m-1}\{p^r\} \longrightarrow F \xrightarrow{f} S^{2m-1}\{p^{r+t}\}$$

is homotopic to the map

$$\begin{aligned}
 & S^{2m-1}\{p^r\} \times \Omega S^{2m-1}\{p^r\} \\
 & \xrightarrow{\rho^t \times \delta_t} S^{2m-1}\{p^{r+t}\} \times S^{2m-1}\{p^{r+t}\} \xrightarrow{\mu} S^{2m-1}\{p^{r+t}\}
 \end{aligned}$$

where  $\delta_t$  is the composition

$$\Omega S^{2m-1}\{p^r\} \xrightarrow{\Omega\pi} \Omega S^{2m-1} \xrightarrow{\iota} S^{2m-1}\{p^{r+t}\}.$$

Since  $\beta$  has order  $p^r$ , it factors through  $F$  and we conclude that  $\beta$  is homotopic to a composition:

$$\begin{aligned}
 \Sigma X & \longrightarrow S^{2m-1}\{p^r\} \times \Omega S^{2m-1}\{p^r\} \\
 & \xrightarrow{\rho^t \times \delta_t} S^{2m-1}\{p^{r+t}\} \times S^{2m-1}\{p^{r+t}\} \longrightarrow S^{2m-1}\{p^{r+t}\}.
 \end{aligned}$$

This map is homotopic to the sum of the two compositions

$$\begin{aligned}
 \Sigma X & \longrightarrow S^{2m-1}\{p^r\} \xrightarrow{\rho^t} S^{2m-1}\{p^{r+t}\} \\
 \Sigma X & \longrightarrow \Omega S^{2m-1}\{p^r\} \longrightarrow \Omega S^{2m-1} \longrightarrow \Omega P^{2m-1}\{p^{r+t}\}.
 \end{aligned}$$

By the naturality of  $\Phi$ ,  $\Phi\beta$  is homotopic to the sum of the maps

$$\begin{aligned}\Sigma X &\xrightarrow{\tilde{\varphi}_1} \Omega P^{2m}(p^r) \xrightarrow{\Omega\rho^t} \Omega P^{2m}(p^{r+t}) \\ \Sigma X &\xrightarrow{\tilde{\varphi}_2} \Omega S^{2m-1} \xrightarrow{\Omega\iota_{2m-1}} \Omega P^{2m}(p^{r+t}).\end{aligned}$$

Thus  $\varphi \equiv \tilde{\phi}\beta$  which is homotopic to  $\varphi_1\rho^t + \iota_{2m-1}\varphi_2$ .  $\square$

At this point we will examine the effects of  $H$ -space based and relative Whitehead products on congruence classes. We recall the Whitehead products in mod  $p^r$  homotopy introduced by Neisendorfer ([Nei80], [Nei10a], [Nei10b]). All coefficient groups will be  $Z/p^r$  with the understanding that in case of different coefficient groups we reduce the larger coefficients to the smaller as in (6.6).

Suppose then we are given a principal fibration

$$\Omega T \xrightarrow{i} E \xrightarrow{\pi} B$$

induced by a map  $\varphi: B \rightarrow T$  where  $T$  is a homotopy commutative  $H$ -space with strict unit. Suppose we are given classes

$$\alpha \in \pi_m(B; Z/p^r), \beta \in \pi_n(B; Z/p^r), \gamma \in \pi_k(E; Z/p^r), \delta \in \pi_\ell(E; Z/p^r).$$

Recall that by using the map  $\Delta$  from section 6 we define the  $H$ -space based Whitehead product

$$[\alpha, \beta]_\times = \{\alpha, \beta\}_\times \Delta \in \pi_{m+n-1}(E; Z/p^r)$$

and the relative Whitehead product

$$[\alpha, \gamma]_r = \{\alpha, \gamma\}_r \Delta \in \pi_{m+k-1}(E; Z/p^r).$$

These are related as in 3.6, 3.10 and 3.11.

**Proposition 7.7.** (a)  $\pi_*[\alpha, \beta]_\times = [\alpha, \beta] \in \pi_{m+n-1}(B; Z/p^r)$

$$(b) [\pi_*\gamma, \pi_*\delta]_\times = [\gamma, \delta] \in \pi_{k+\ell-1}(E; Z/p^r)$$

$$(c) [\alpha, \delta]_r = [\alpha, \pi_*\delta]_\times \in \pi_{m+\ell-1}(E; Z/p^r)$$

$$(d) \pi_*[\alpha, \delta]_r = [\alpha, \pi_*\delta] \in \pi_{m+\ell-1}(B; Z/p^r)$$

$$(e) [\pi_*\gamma, \delta]_r = [\gamma, \delta] \in \pi_{k+\ell-1}(E; Z/p^r)$$

According to Neisendorfer [Nei10a], we also have standard Whitehead product formulas:

**Proposition 7.8.** *The following identities hold:*

$$(a) [\alpha, \beta]_\times = -(-1)^{(m+1)(n+1)}[\beta, \alpha]_\times$$

$$(b) [\alpha_1 + \alpha_2, \beta]_\times = [\alpha_1, \beta]_\times + [\alpha_2, \beta]_\times$$

$$(c) [\alpha, [\beta, \eta]]_\times = [[\alpha, \beta], \eta]_\times + (-1)^{(m+1)(n+1)}[\beta, [\alpha, \eta]]_\times \text{ for } \eta \in \pi_j(B; Z/p^r)$$

$$(d) \beta^{(r)}[\alpha, \beta]_\times = [\beta^{(r)}\alpha, \beta]_\times + (-1)^{m+1}[\alpha, \beta^{(r)}\beta]_\times \text{ where } \beta^{(r)} \text{ is the Bockstein associated with the composition } P^k(p^r) \rightarrow P^{k+1}(p^r) \text{ for appropriate } k.$$

*Proof.* See [Nei10a]. Neisendorfer considers the adjoint Samelson products, so there is a dimension shift.  $\square$

We now define a  $Z/p^r$  module  $M_*(B)$  by

$$M_m = \pi_{m+1}(B; Z/p^r).$$

**Proposition 7.9.** *The relative Whitehead product induces a homomorphism:*

$$M_m \otimes e\pi_k(E; Z/p^r) \rightarrow e\pi_{m+k}(E; Z/p^r).$$

*Proof.* It suffices to show that if  $\Sigma\gamma \sim *$ , then  $\Sigma[\alpha, \gamma]_r \sim *$ . Recall that  $[\alpha, \gamma]_r$  is given in 3.9 by the composition:

$$\begin{aligned} P^{m+k}(p^r) &\xrightarrow{\Delta} P^m(p^r) \wedge P^k(p^r) \\ &\xrightarrow{\theta} \Omega P^{m+1}(p^r) \ltimes P^k(p^r) \xrightarrow{\Omega\alpha \ltimes \gamma} \Omega B \ltimes E \xrightarrow{\Gamma'} E. \end{aligned}$$

However, since there is a natural homeomorphism

$$\Sigma(X \ltimes Y) \cong X \ltimes \Sigma Y,$$

$\Sigma[\alpha, \gamma]_r$  factors through the map

$$\Omega P^{m+1} \ltimes P^{k+1}(p^r) \xrightarrow{\Omega\alpha \ltimes \Sigma\gamma} \Omega B \ltimes \Sigma E.$$

This map is null homotopic since it factors through  $\Omega B \ltimes *$  up to homotopy.  $\square$

**Definition 7.10.**  $A_*(B)$  is the graded symmetric algebra generated by  $M_*(B)$ .

**Theorem 7.11.** *The relative Whitehead product induces the structure of a graded differential  $A_*(B)$  module on  $e\pi_*(E; Z/p^r)$ .*

*Proof.* By 7.9, there is an action of  $M_*(A)$  on  $e\pi_*(E; Z/p^r)$  and hence an action of the tensor algebra. It suffices to show that

$$[\alpha, [\beta, \gamma]_r]_r \equiv (-1)^{mn} [\beta, [\alpha, \gamma]_r]_r$$

where  $\alpha \in M_m$  and  $\beta \in M_n$ . However we have

$$\begin{aligned} [\alpha, [\beta, \gamma]_r]_r &= [\alpha, [\beta, \pi_*(\gamma)]]_{\times} \\ &= [[\alpha, \beta], \pi_*(\gamma)]_{\times} + (-1)^{mn} [\beta, [\alpha, \pi_*(\gamma)]]_{\times} \end{aligned}$$

by 7.7(c), (d) and 7.8(c). But  $[\alpha, \beta] = \pi_*[\alpha, \beta]_{\times}$  by 7.7(a), so we have

$$\begin{aligned} [[\alpha, \beta], \pi_*(\gamma)]_{\times} &= [\pi_*[\alpha, \beta]_{\times}, \pi_*\gamma]_{\times} \\ &= [[\alpha, \beta]_{\times}, \gamma] \end{aligned}$$

by 7.7(b). But  $[[\alpha, \beta]_{\times}, \gamma]$  is a Whitehead product of classes in  $\pi_*(E; Z/p^r)$ , so  $[[\alpha, \beta]_{\times}, \gamma] \equiv 0$ . Thus

$$\begin{aligned} [\alpha, [\beta, \gamma]_r]_r &\equiv (-1)^{mn} [\beta, [\alpha, \pi_*\gamma]]_{\times} \\ &= (-1)^{mn} [\beta, [\alpha, \gamma]_r]_r. \end{aligned} \quad \square$$

**Proposition 7.12.** *The action of  $A_*(B)$  on  $e\pi_*(E; Z/p^r)$  commutes with the composition from 7.4. That is, given  $\xi: P^n \rightarrow P^\ell$ ,  $\alpha: P^{m+1} \rightarrow B$ , and  $\gamma: P^\ell \rightarrow E$ , we have*

$$[\alpha, \gamma\xi]_r \equiv [\alpha, \gamma]_r \Sigma^m \xi.$$

**Remark.** In the case of the absolute Whitehead products, one only has a homotopy

$$[\alpha, \gamma\xi] \sim [\alpha, \gamma]\Sigma^m \xi$$

when  $\xi$  is a co- $H$  map. This is a special feature of the relative Whitehead product and the congruence relation.

*Proof.* The two homotopy classes in question are represented by the following compositions

$$\begin{aligned} P^{m+n} &\xrightarrow{\Sigma^m \xi} P^{m+\ell} \xrightarrow{\Delta} P^m \wedge P^\ell \xrightarrow{\theta} \Omega P^{m+1} \ltimes P^\ell \xrightarrow{\Omega\alpha \ltimes \gamma} \Omega B \ltimes E \xrightarrow{\Gamma'} E \\ P^{m+n} &\xrightarrow{\Delta} P^m \wedge P^n \xrightarrow{\theta} \Omega P^{m+1} \ltimes P^n \xrightarrow{\Omega\alpha \ltimes \gamma\xi} \Omega B \ltimes E \xrightarrow{\Gamma'} E. \end{aligned}$$

Thus it suffices to show that the square

$$\begin{array}{ccc} P^m \wedge P^\ell & \xrightarrow{\theta} & \Omega P^{m+1} \ltimes P^\ell \\ \uparrow 1 \wedge \xi & \equiv & \uparrow 1 \ltimes \xi \\ P^m \wedge P^n & \xrightarrow{\theta} & \Omega P^{m+1} \ltimes P^n \end{array}$$

commutes up to congruence. The result follows from

**Lemma 7.13.** *For any map  $\xi: \Sigma Y \rightarrow \Sigma Y'$ , the following square commutes up to congruence:*

$$\begin{array}{ccc} X \wedge \Sigma Y \simeq X * Y & \xrightarrow{\theta} & \Omega \Sigma X \ltimes \Sigma Y \\ \downarrow 1 \wedge \xi & \equiv & \downarrow 1 \ltimes \xi \\ X \wedge \Sigma Y' \simeq X * Y' & \xrightarrow{\theta} & \Omega \Sigma X \ltimes \Sigma Y' \end{array}$$

*Proof.* The map  $\theta$  from 3.9 is a lifting of the Whitehead product

$$X * Y \xrightarrow{W} \Sigma X \vee \Sigma Y.$$

to the homotopy fiber of the projection  $\Sigma X \vee \Sigma Y \longrightarrow \Sigma X$ .  $\theta$  is unique up to homotopy such that the diagram

$$\begin{array}{ccc} & \Omega \Sigma X \ltimes \Sigma Y & \\ \theta \nearrow & \downarrow \pi & \\ X * Y & \xrightarrow{W} & \Sigma X \vee \Sigma Y \end{array}$$

commutes up to homotopy.

The map  $\pi: \Omega \Sigma X \ltimes \Sigma Y \rightarrow \Sigma X \vee \Sigma Y$  factors, up to homotopy

$$\Omega \Sigma X \ltimes \Sigma Y \simeq P \Sigma X \cup \Omega \Sigma X \times \Sigma Y \xrightarrow{\pi'} \Sigma Y \vee \Sigma Y$$

where  $\pi'$  is end point projection on  $P\Sigma X$  and on  $\Omega\Sigma X \times \Sigma Y$  it is projection onto the second coordinate. Consequently, a choice for  $\theta$  factors

$$X * Y \xrightarrow{\theta'} X \ltimes \Sigma Y \xrightarrow{\iota \ltimes 1} \Omega\Sigma X \ltimes \Sigma Y$$

where  $\theta'$  maps  $CX \times Y \subset X * Y$  to the base point and  $\theta'$  maps  $X \times CY$  to  $X \times \Sigma Y \rightarrow X \ltimes \Sigma Y$  by pinching  $X \times Y$  to a point. In fact,  $\theta'$  is the quotient map:

$$X * Y \rightarrow X * Y / CX \times Y \cong X \ltimes \Sigma Y.$$

The subspace  $CX \times Y$  is homotopy equivalent to  $Y$  and the inclusion:

$$CX \times Y \subset X * Y$$

is null homotopic. Consequently we have a split cofibration sequence:

$$X * Y \xrightarrow{\theta'} X \ltimes \Sigma Y \longrightarrow \Sigma Y.$$

Using naturality of the projection  $X \ltimes \Sigma Y \longrightarrow \Sigma Y$  for map  $\xi: \Sigma Y \rightarrow \Sigma Y'$ , we construct a diagram:

$$(7.14) \quad \begin{array}{ccccc} X * Y & \xrightarrow{\theta} & X \ltimes \Sigma Y & \longrightarrow & \Sigma Y \\ \xi' \downarrow & & \equiv \iota \ltimes \xi \downarrow & & \xi \downarrow \\ X * Y' & \longrightarrow & X \ltimes \Sigma Y' & \longrightarrow & \Sigma Y'. \end{array}$$

We assert that there is a map  $\xi'$  so that the lefthand square commutes up to congruence.<sup>8</sup> This follows since  $X \ltimes \Sigma Y' \simeq X * Y' \vee \Sigma Y'$  using 7.3. We next observe that the composition

$$X * Y \xrightarrow{\theta} X \ltimes \Sigma Y \longrightarrow X \wedge \Sigma Y$$

obtained by pinching  $\Sigma Y$  to a point, is a homotopy equivalence. Thus the diagram

$$\begin{array}{ccccc} X * Y & \xrightarrow{\theta} & X \ltimes \Sigma Y & \longrightarrow & X \wedge \Sigma Y \\ \xi' \downarrow & & \iota \ltimes \xi \downarrow & & \iota \wedge \xi \downarrow \\ X * Y' & \xrightarrow{\theta} & X \ltimes \Sigma Y' & \longrightarrow & X \wedge \Sigma Y' \end{array}$$

commutes up to congruence. Inverting the equivalence, we get a square which commutes up to congruence

$$\begin{array}{ccc} X \wedge \Sigma Y & \xrightarrow{\simeq} & X * Y \\ \iota \wedge \xi \downarrow & \equiv & \xi' \downarrow \\ X \wedge \Sigma Y' & \xrightarrow{\simeq} & X * Y' \end{array}$$

combining this with (7.14) proves the lemma and the theorem.  $\square$

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<sup>8</sup>This definitely does not commute up to homotopy in general.

We now apply the  $A_*(D_k)$  module structure to the study of the congruence classes of the obstructions in 6.13. We will actually only consider the subalgebra of  $A_*(D_k)$  generated by  $\nu \in M_{2n} = \pi_{2n+1}(D_k Z/p_r)$  and  $\mu = \beta\nu \in M_{2n-1}$ . These elements generate a subalgebra

$$Z/p[\nu] \otimes \wedge(\mu) \subset A_*(D_k)$$

and  $e\pi_*(J_k; Z/p^r)$  is a module over this algebra. We define classes

$$\begin{aligned} \overline{a(i)} &= \tau_k a(i) \rho^{i-1} : P^{2np^i} \rightarrow J_k \\ \overline{b(i)} &= \tau_k a(i) \delta_{i-1} : P^{2np^i-1} \rightarrow J_k \\ \overline{a(0)} &= [\nu, \mu]_{\times} \\ \overline{b(0)} &= [\mu, \mu]_{\times} \end{aligned}$$

for  $1 \leq i \leq k$ .

**Theorem 7.15.** *The collection of congruence classes of the set of obstructions listed in 6.13 is spanned, as a module over  $Z/p[\nu] \otimes \wedge(\mu)$  by the classes  $\overline{a(i)}$  and  $\overline{b(i)}$  of weight  $j \geq 2$  for  $0 \leq i \leq k$ .*

*Proof.* We first consider internal  $H$ -space based Whitehead products of weight 2. Recall that by 7.7(b)  $[\xi_k \gamma, \xi_k \delta]_{\times} = [\gamma, \delta]$  which is null congruent, so we need only consider weight 2 products  $[x_1, x_2]_{\times}$  in which at least one of  $x_1, x_2$  is not in the image of  $\xi_k$ . By 7.8(a), we will assume that  $x_1 = \mu$  or  $\nu$ . This gives the following possibilities for weight 2.

$$\overline{a(0)}, \overline{b(0)}, [\nu, \xi_k \overline{a(i)}]_{\times}, [\mu, \xi_k \overline{a(i)}]_{\times}, [\nu, \xi_k \overline{b(i)}]_{\times}, [\mu, \xi_k \overline{b(i)}]_{\times}$$

for  $1 \leq i \leq k$ . Applying 7.7(b) again, we see that for  $j > 2$  the class of  $[x_1, \dots, x_j]_{\times}$  is null congruent if  $x_1$  is in the image of  $\xi_k$ . Thus each of  $x_1, \dots, x_{j-2}$  must be either  $\mu$  or  $\nu$ . Furthermore, for  $j > 2$

$$\begin{aligned} [x_1, \dots, x_j]_{\times} &= [x_1, \xi_k [x_2, \dots, x_j]]_{\times} \\ &= [x_1, [x_2, \dots, x_j]]_r \end{aligned}$$

by 3.10, so  $[x_1, \dots, x_j]_{\times}$  is in the  $Z/p[\nu] \otimes \wedge(\mu)$  submodule generated by  $[x_{j-1}, x_j]_{\times}$ .  $\square$

We will refer to the submodule generated by  $\overline{a(k)}$  and  $\overline{b(k)}$  as the level  $k$  obstructions. In case  $k = 0$  we have some simple relations:

**Proposition 7.16.**  *$\overline{\mu b(0)} \equiv 0$  and  $\nu \overline{b(0)} \equiv 2\mu \overline{a(0)}$ . Consequently the submodule generated by  $\overline{a(0)}$  and  $\overline{b(0)}$  has a basis consisting of  $\nu^k \overline{b(0)}$  and  $\nu^k \overline{a(0)}$  for  $k \geq 0$ .*

*Proof.*  $\mu[\mu, u]_{\times} \equiv [\mu, [\mu, u]_{\times}]_r \equiv [\mu, [\mu, u]]_{\times} \equiv 0$ .  $\mu[\nu, \mu]_{\times} \equiv [\mu, [\nu, \mu]_{\times}]_r \equiv [\mu, [\nu, \mu]]_{\times}$ . Using 7.8(a) and (c) we get  $[\mu, [\nu, \mu]]_{\times} = [[\mu, \nu], \mu]_{\times} + [\nu, [\mu, \mu]]_{\times} = -[\mu, [\nu, \mu]]_{\times} + [\nu, [\mu, \mu]]_{\times}$ , so  $2\mu[\nu, \mu]_{\times} = [\nu, [\mu, \mu]]_{\times} \equiv \nu[\mu, \mu]_{\times}$ .  $\square$

Because of these relations we define<sup>9</sup>  $x_2 = [\nu, \mu]_\times$  and  $y_2 = \frac{1}{2}[\mu, \mu]_\times$ . Then  $x_k = \nu x_{k-1} = \nu^{k-2} \overline{a(0)}$  and  $y_k = \mu x_{k-1}$ .

**Proposition 7.17.** *The level 0 congruence classes are generated by  $x_j$  and  $y_j$  for  $j \geq 2$  with the relations  $\mu x_k \equiv \nu y_k$  and  $\nu y_k \equiv 0$ . Furthermore  $\beta x_j \equiv j y_j$  and  $\beta y_j \equiv 0$ .*

*Proof.*  $\mu x_k \equiv \nu^{k-2} \mu x_2 \equiv \frac{1}{2} \nu^{k-1} [\mu, \mu]_\times \equiv \nu^{k-1} y_2 = \nu y_k$ .  $\mu y_k \equiv \mu \nu^{k-2} y_2 \equiv \frac{1}{2} \nu^{k-2} \mu [\mu, \mu]_\times \equiv 0$  by 7.16. These relations imply that the  $x_k$  and  $y_k$  are linear generators. We will see in section 8 that they are actually linearly independent.  $\beta x_k = (k-2) \nu^{k-3} x_2 + \nu^{k-2} [\mu, \mu]_\times \equiv (k-2) \mu x_{k-1} + 2 y_k \equiv k y_k$ .  $\beta y_k \equiv (k-2) \nu^{k-3} \mu [\mu, \mu]_\times \equiv 0$ .  $\square$

## 8. CONTROLLED EXTENSION

In this section we will introduce the controlled extension theorem and apply it to the simplest case: the construction of a retraction map  $\nu_0: E_0 \rightarrow BW_n$  such that the composition

$$\Sigma(\Omega G_0 \wedge \Omega G_0) \xrightarrow{\Gamma_0} E_0 \xrightarrow{\nu_0} BW_n$$

is null homotopic. This case is considerably simpler than the case  $k > 0$  and will serve as a model for the later cases. The controlled extension theorem is an enhancement of the extension theorem ([GT10, 2.2]), and our construction of  $\nu_0$  is a more controlled version of the construction of  $\nu_0 = \nu^E$  of section 3 of [GT10].

**Theorem 8.1** (Controlled extension theorem). *Suppose that all spaces are localized at  $p > 2$  and we have a diagram of principal fibrations induced by a map  $\varphi: B \cup_\theta e^m \rightarrow X$ :*

$$\begin{array}{ccc} \Omega X & \xlongequal{\quad} & \Omega X \\ \downarrow & & \downarrow \\ E_0 & \xrightarrow{\quad} & E \\ \downarrow & & \downarrow \pi \\ B & \xrightarrow{\quad} & B \cup_\theta e^m. \end{array}$$

*Suppose  $\dim B < m$  and we are given a map  $\chi: P^m(p^s) \rightarrow E$  with  $s \geq 1$  such that  $\pi\chi: P^m(p^s) \rightarrow B \cup_\theta e^m$  induces an isomorphism in mod  $p$  cohomology in dimension  $m$ . Suppose also that we are given a map  $\gamma_0: E_0 \rightarrow BW_n$ . Then:*

- (a) *There is an extension of  $\gamma_0$  to  $\gamma': E \rightarrow BW_n$ .*

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<sup>9</sup>The class  $x_k: P^{2nk} \rightarrow D_0 = P^{2n+1}$  is the adjoint of the similarly named class in [CMN79c].



- (b) Suppose also that we are given a map  $u: P \rightarrow E$  and a subspace  $P_0 \subset P$  such that the composition

$$P_0 \longrightarrow P \xrightarrow{u} E \xrightarrow{\gamma'} BW_n$$

is null homotopic and such that the quotient map  $q: P \rightarrow P/P_0$  factors up to homotopy

$$P \xrightarrow{u} E \xrightarrow{q'} E/E_0 \xrightarrow{\xi} P/P_0$$

for some map  $\xi$ . Then there is an extension  $\gamma$  of  $\gamma_0$  such that  $\gamma u \sim *$ .

*Proof.* The existence of  $\gamma'$  is the extension theorem of [GT10, 2.1]. Basically, the existence of  $\chi$  guarantees that  $\theta$  is divisible by  $p^s$  and this occurs in  $E_0$ . A clutching construction model for  $E$  is introduced and  $E$  is described as a pushout. The fact that the  $H$ -space exponent of  $BW_n$  is  $p$  is used to construct  $\gamma_0$ .

To prove part (b), we suppose  $\gamma'$  is given and we construct  $\gamma$  as the composition:

$$E \xrightarrow{\Delta} E \times E/E_0 \xrightarrow{\gamma' \times \xi} BW_n \times P/P_0 \xrightarrow{1 \times \eta} BW_n \times BW_n \xrightarrow{\div} BW_n$$

where  $\eta: P/P_0 \rightarrow BW_n$  is defined by the null homotopy of  $\gamma' u|_{P_0}$  and  $\div$  is the  $H$ -space division map. Clearly  $\gamma|_{E_0} \sim \gamma_0$ . To study  $\gamma|_P$ , consider the diagram

$$\begin{array}{ccccccc} E & \xrightarrow{\Delta} & E \times E/E_0 & \xrightarrow{\gamma' \times \xi} & BW_n \times P/P_0 & \xrightarrow{1 \times \eta} & BW_n \times BW_n \xrightarrow{\div} BW_n \\ \uparrow u & & \uparrow u \times q' u & & \uparrow & & \parallel \\ P & \xrightarrow{\Delta} & P \times P & \xrightarrow{\eta q \times q} & BW_n \times P/P_0 & \xrightarrow{1 \times \eta} & BW_n \times BW_n \xrightarrow{\div} BW_n \end{array}$$

where the lower composition of the first 3 maps factors through the diagonal map of  $BW_n$ , so the lower composition is null homotopic.  $\square$

Now we will apply this in the case  $k = 0$ . Recall the spaces (6.2). In this case  $D_0 = G_0 = P^{2n+1}$  and  $J_0 = E_0$ :

$$\begin{array}{ccccc} \Omega^2 S^{2n+1} & \longrightarrow & E_0 & \xrightarrow{\tau_0} & F_0 & \longrightarrow & \Omega S^{2n+1} \\ & & \downarrow & & \downarrow & & \downarrow \\ & & P^{2n+1} & \xlongequal{\quad} & P^{2n+1} & \longrightarrow & P S^{2n+1} \\ & & \downarrow & & \downarrow & & \downarrow \\ & & S^{2n+1}\{p^r\} & \longrightarrow & S^{2n+1} & \xrightarrow{p^r} & S^{2n+1}. \end{array}$$

These spaces were introduced in [CMN79a] where  $E_0$  is called  $E^{2n+1}(p^r)$  and  $F_0$  is called  $F^{2n+1}(p^r)$ .

**Proposition 8.2.**  $H^i(F_0; Z_{(p)}) = \begin{cases} Z_{(p)} & \text{if } i = 2mn \\ 0 & \text{otherwise.} \end{cases}$

*Proof.* This is immediate from consideration of the cohomology Serre spectral sequence of the middle fibration, which is induced from the path space fibration on the right. All differentials are controlled by the path space fibration.  $\square$

We filter  $F_0$  by setting  $F_0(m)$  to be the  $2mn$  skeleton of  $F_0$  and define  $E_0(m)$  to be the pullback over  $F_0(m)$

$$\begin{array}{ccc} \Omega^2 S^{2n+1} & \xlongequal{\quad} & \Omega^2 S^{2n+1} \\ \downarrow & & \downarrow \\ E_0(m) & \xrightarrow{\quad} & E_0 \\ \downarrow & & \downarrow \eta_0 \\ F_0(m) & \xrightarrow{\quad} & F_0 \end{array}$$

Since  $F_0(1) = S^{2n}$ , this fibration in case  $m = 1$  is the fibration which defines  $BW_n$  ([Gra88])

$$\Omega^2 S^{2n+1} \longrightarrow S^{4n-1} \times BW_n \longrightarrow S^{2n} \longrightarrow \Omega S^{2n+1}.$$

We consequently define  $\nu_0(1): E_0(1) \rightarrow BW_n$  by retracting onto  $BW_n$ . Clearly  $\nu_0(1)y_2$  is null homotopic since  $y_2: P^{4n-1} \rightarrow E_0$  and  $BW_n$  is  $2np-3$  connected. We will use 8.1 to construct  $\nu_0(m): E_0(m) \rightarrow BW_n$  such that  $\nu_0(m)y_i$  and  $\nu_0(m)x_{i-1}$  are null homotopic for  $i \leq m+1$ .

**Proposition 8.3.** *For each  $m \geq 2$ , there is a retraction  $\nu_0(m): E_0(m) \rightarrow BW_n$  extending  $\nu_0(m-1)$  such that  $\nu_0(m)_*$  annihilates  $x_m$  and  $y_{m+1}$ .*

The proof of this result will depend on two lemmas.

**Lemma 8.4.** *The composition*

$$P^{2nj} \xrightarrow{x_j} E_0 \xrightarrow{\eta_0} F_0$$

*induces a cohomology epimorphism.*

*Proof.* To study  $\eta_0 x_j$ , we use the principal fibration:

$$\Omega S^{2n+1} \longrightarrow F_0 \longrightarrow P^{2n+1}$$

Clearly  $\mu: P^{2n} \rightarrow P^{2n+1}$  lifts to a map  $x_1: P^{2n} \rightarrow F_0$  which induces a cohomology epimorphism. Then  $x_j = [\nu, x_{j-1}]_r$  for each  $j \geq 2$  so we can apply 3.12 to evaluate  $x_j$  in cohomology.  $\square$

**Lemma 8.5.** *The composition*

$$\begin{aligned} p^{2n(j+1)-1} &\xrightarrow{y_{j+1}} E_0(j) \longrightarrow E_0(j)/E_0(j-1) \\ &\simeq S^{2nj} \rtimes \Omega^2 S^{2n+1} \xrightarrow{\xi} S^{2nj} \vee S^{2n(j+1)-1} \end{aligned}$$

*induces an integral cohomology epimorphism.*

*Proof.* Since  $F_0(j) = F_0(j-1) \cup e^{2mj}$ ,  $E_0(j)/E_0(j-1) \simeq S^{2mj} \rtimes \Omega^2 S^{2n+1}$  by the clutching construction ([Gra88]). From the homotopy commutative square

$$\begin{array}{ccc} E_0(j) & \longrightarrow & E_0(j)/E_0(j-1) \simeq S^{2nj} \rtimes \Omega^2 S^{2n+1} \\ \downarrow & & \downarrow \\ F_0(j) & \longrightarrow & S^{2nj} \end{array}$$

we see that the composition

$$\begin{aligned} P^{2nj} &\xrightarrow{x_j} E_0(j) \longrightarrow E_0(j)/E_0(j-1) \\ &\simeq S^{2nj} \rtimes \Omega^2 S^{2n+1} \xrightarrow{\xi} S^{2nj} \vee S^{2n(j+1)-1} \end{aligned}$$

is an integral cohomology epimorphism. Since the action of  $\Omega^2 S^{2n+1}$  on  $E_0(j)$  corresponds with the action of  $\Omega^2 S^{2n+1}$  on  $E_0(j)/E_0(j-1) \simeq S^{2nj} \rtimes \Omega^2 S^{2n+1}$ , we can apply 3.12 to see that the composition in question

$$P^{2n(j+1)-1} \longrightarrow S^{2nj} \rtimes \Omega^2 S^{2n+1}$$

induces an epimorphism in integral cohomology.  $\square$

*Proof of 8.3.* We apply 8.1 to the diagram

$$\begin{array}{ccc} \Omega^2 S^{2n+1} & \xlongequal{\quad} & \Omega^2 S^{2n+1} \\ \downarrow & & \downarrow \\ E_0(m-1) & \longrightarrow & E_0(m) \\ \downarrow & & \downarrow \\ F_0(m-1) & \longrightarrow & F_0(m) = F_0(m-1) \cup_{\theta} e^{2mn} \end{array}$$

where  $\theta$  is the attaching map of the  $2mn$  cell. Let  $\chi = x_m: P^{2mn} \rightarrow E(m)$ . Choose an extension  $\gamma'$  of  $\gamma_0 = \nu_0(m-1)$ . Let  $P = P^{2mn} \vee P^{2(m+1)n-1}$  and

$$u: P^{2mn} \vee P^{2(m+1)n-1} \xrightarrow{x_m \vee y_{m+1}} E_0(m)$$

and  $P_0 = S^{2mn-1} \vee S^{2(m+1)n-2}$ . The composition

$$P^{2mn-1} \longrightarrow S^{2mn-1} \longrightarrow P^{2mn} \xrightarrow{x_m} E_0(m)$$

is  $\beta x_m \equiv my_m$  by 7.17. Since  $\nu_0(m-1)y_m$  is null homotopic, the composition

$$P^{2mn-1} \xrightarrow{\beta} P^{2mn} \xrightarrow{x_m} E_0(m) \xrightarrow{\gamma'} BW_n$$

is null homotopic.  $\beta$  factors:  $P^{2mn-1} \rightarrow S^{2mn-1} \rightarrow P^{2mn}$ , so the composition

$$S^{2mn-1} \longrightarrow P^{2mn} \xrightarrow{x_m} E_0(m) \xrightarrow{\gamma'} BW_n$$

is divisible by  $p^r$ . However  $p \cdot \pi_*(BW_n) = 0$ , so this composition is null homotopic. Similarly, since  $\beta y_{m+1} \equiv 0$ , the composition

$$S^{2(m+1)n-2} \longrightarrow P^{2(m+1)n-1} \xrightarrow{y_{m+1}} E_0(m) \xrightarrow{\gamma'} BW_n$$

is null homotopic. Thus the composition

$$P_0 \longrightarrow P \xrightarrow{x_m \vee y_{m+1}} E_0(m) \xrightarrow{\gamma'} BW_n$$

is null homotopic. However

$$P = P^{2mn} \vee P^{2(m+1)n-1} \xrightarrow{x_m \vee y_{m+1}} E_0(m) \longrightarrow E_0(m)/E_0(m-1) \cong S^{2mn} \rtimes \Omega^2 S^{2n+1}$$

induces an integral cohomology epimorphism by 8.4 and 8.5. Let  $\xi$  be the composition

$$S^{2mn} \rtimes \Omega^2 S^{2n+1} \simeq S^{2mn} \vee S^{2mn} \wedge \Omega^2 S^{2n+1} \longrightarrow S^{2mn} \vee S^{2(m+1)n-1}$$

where the last map is obtained by evaluation on the double loop space. Clearly the conditions of 8.1 are satisfied, so we can choose an extension  $\nu_0(m)$  of  $\nu_0(m-1)$  such that  $\nu_0(m)_*(x_m) = 0$  and  $\nu_0(m)_*(y_{m+1}) = 0$ . This completes the induction.  $\square$

**Corollary 8.6.** *There is a retraction  $\nu_0: E_0 \rightarrow BW_n$  such that the composition*

$$\Sigma(\Omega G_0 \wedge \Omega G_0) \xrightarrow{\Gamma_0} E_0 \xrightarrow{\nu_0} BW_n$$

*is null homotopic.*

*Proof.* By 8.3  $(\nu_0)_*(x_m) = 0$  and  $(\nu_0)_*(y_m) = 0$  for  $m \geq 2$ . By 6.13 and 7.17,

$$ad_{\times}^j: \Sigma G_0 \wedge \cdots \wedge G_0 \rightarrow E_0 \xrightarrow{\nu_0} BW_n$$

is null homotopic for all  $j \geq 2$ . The conclusion follows from 6.7.  $\square$

**Remark 8.7.** The map  $\nu_0$  is a specific choice of the map  $\nu^E$  asserted to exist in [GT10, 3.5]. Consequently by [GT10, 3.10] and [GT10, 4.9], the space  $T_{2n-1}$  constructed as the fiber of  $\nu_0$  is homotopy equivalent to the space constructed in [GT10] for any of the possible choices of  $\nu^E$  made there.

## 9. INTERLUDE

At this point we pause in the induction. We suppose that we have constructed a retraction

$$\gamma_{k-1}: J_{k-1} \rightarrow BW_n$$

such that the composition

$$\Sigma(\Omega G_{k-1} \wedge \Omega G_{k-1}) \xrightarrow{\Gamma_{k-1}} E_{k-1} \xrightarrow{\tau_{k-1}} J_{k-1} \xrightarrow{\gamma_{k-1}} BW_n$$

is null homotopic. The procedure in section 8 is a model for the inductive step. To proceed, we will first need to prove:

$$(9.7) \quad H_i(F_k; Z_{(p)}) \cong \begin{cases} Z_{(p)} & \text{if } i = 2mn \\ 0 & \text{otherwise.} \end{cases}$$

This will allow an inductive procedure over the skeleta of  $F_k$  as in section 8. However, a difficulty arises because an extension over  $J_k$  which annihilates the level  $k$  obstruction may not necessarily be an extension of  $\gamma_{k-1}$ . We will need to find a relationship between the level  $k$  obstructions and the earlier ones to show that all obstructions of level less than or equal to  $k$  are annihilated by  $\gamma_k$ . This will be established in section 10.

**Proposition 9.1.** *Let  $W_{k-1}$  be the fiber of  $\gamma_{k-1}$ . Then we have a homotopy commutative diagram of vertical fibration sequences*

$$\begin{array}{ccccc} T & \xlongequal{\quad} & T & \longrightarrow & \Omega S^{2n+1} \\ \downarrow & & \downarrow & & \downarrow \\ R_{k-1} & \longrightarrow & W_{k-1} & \longrightarrow & F_{k-1} \\ \downarrow & & \downarrow & & \downarrow \sigma_{k-1} \\ G_{k-1} & \longrightarrow & D_{k-1} & \xlongequal{\quad} & D_{k-1} \end{array}$$

and two diagrams of fibration sequences

$$\begin{array}{ccccc}
 S^{2n-1} & \longrightarrow & \Omega^2 S^{2n-1} & \xrightarrow{\nu} & BW_n \\
 \downarrow & & \downarrow & & \parallel \\
 W_{k-1} & \longrightarrow & J_{k-1} & \xrightarrow{\gamma_{k-1}} & BW_n \\
 \downarrow & & \downarrow & & \\
 F_{k-1} & \xlongequal{\quad} & F_{k-1} & & 
 \end{array}$$

$$\begin{array}{ccccc}
 S^{2n-1} & \longrightarrow & T & \longrightarrow & \Omega S^{2n+1} \\
 \downarrow & & \downarrow & & \\
 W_{k-1} & \xlongequal{\quad} & W_{k-1} & & \\
 \downarrow & & \downarrow & & \\
 F_{k-1} & \longrightarrow & D_{k-1} & \longrightarrow & S^{2n+1}
 \end{array}$$

*Proof.* The map  $\nu_{k-1}: E_{k-1} \rightarrow BW_n$  defined in [GT10, 4.3(h)] was an arbitrary retraction such that the composition

$$\Omega^2 S^{2n+1} \longrightarrow \Omega S^{2n+1}\{p^r\} \longrightarrow E_0 \longrightarrow E_{k-1} \xrightarrow{\nu_{k-1}} BW_k$$

is homotopic to  $\nu$ . Since the composition of the first two maps factors:

$$\Omega^2 S^{2n-1} \xrightarrow{\nu} BW_n \rightarrow BW_n \times S^{4n-1} = E_o(1) \rightarrow E_0,$$

we can define  $\nu_{k-1}$  to be the composition

$$E_{k-1} \xrightarrow{\tau_{k-1}} J_{k-1} \xrightarrow{\gamma_{k-1}} BW_n.$$

From this it follows that we have a commutative diagram of fibration sequences

$$\begin{array}{ccccc}
 R_{k-1} & \longrightarrow & E_{k-1} & \xrightarrow{\nu_{k-1}} & BW_n \\
 \downarrow & & \downarrow \tau_{k-1} & & \parallel \\
 W_{k-1} & \longrightarrow & J_{k-1} & \xrightarrow{\gamma_{k-1}} & BW_n.
 \end{array}$$

Consequently the square

$$\begin{array}{ccc}
 R_{k-1} & \longrightarrow & W_{k-1} \\
 \downarrow & & \downarrow \\
 G_{k-1} & \longrightarrow & D_{k-1}
 \end{array}$$

is the composition of two pullback squares

$$\begin{array}{ccccc} R_{k-1} & \longrightarrow & E_{k-1} & \xrightarrow{\pi_{k-1}} & G_{k-1} \\ \downarrow & & \downarrow \tau_{k-1} & & \downarrow \\ W_{k-1} & \longrightarrow & J_{k-1} & \xrightarrow{\xi_{k-1}} & D_{k-1}, \end{array}$$

so it is a pullback and the first diagram commutes up to homotopy. The second diagram follows from the definition of  $W_{k-1}$  and the third is a combination of the first two.  $\square$

**Proposition 9.2.**  $\Omega F_{k-1} \simeq S^{2n-1} \times \Omega W_{k-1}$ .

*Proof.* Extending the third diagram of 9.1 to the left yields a diagram

$$\begin{array}{ccc} \Omega^2 S^{2n+1} & \longrightarrow & S^{2n-1} \\ \downarrow & & \downarrow \\ * & \longrightarrow & W_{k-1} \\ \downarrow & & \downarrow \\ \Omega S^{2n+1} & \longrightarrow & F_{k-1}; \end{array}$$

both of the horizontal maps have degree  $p^r$  in dimension  $2n$ , so  $W_{k-1}$  is  $4n-2$  connected and the map  $S^{2n-1} \rightarrow W_{k-1}$  is null homotopic. From this it follows that  $\Omega F_{k-1} \simeq S^{2n-1} \times \Omega W_{k-1}$ .  $\square$

**Proposition 9.3.** *The homomorphism  $H^*(F_{k-1}) \rightarrow H^*(W_{k-1})$  is onto and*

$$H^j(W_{k-1}) = \begin{cases} Z_{(p)}/p^{r+s-1} & \text{if } j = 2np^s \quad 0 < s < k \\ Z_{(p)}/ip^r & \text{if } j = 2ni, \text{ otherwise} \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* Consider the Serre spectral sequence for the  $p$ -local homology of the fibration

$$\Omega S^{2n+1} \xrightarrow{\delta_{k-1}} F_{k-1} \xrightarrow{\sigma_{k-1}} D_{k-1}$$

Since  $E_{p,q}^2$  is only nonzero when  $p$  and  $q$  are divisible by  $2n$ ,  $E_{p,q}^2 = E_{p,q}^\infty$ . We assume the result (9.7) for the case  $k-1$  by induction. Since  $E_{p,q}^\infty$  has finite order when  $p > 0$  and  $H_*(F_{k-1}; Z_{(p)})$  is free, all extensions are nontrivial. Let  $u_i \in H^{2ni}(\Omega S^{2n+1})$  be the generator dual to the  $i^{\text{th}}$  power of a chosen fixed generator in  $H_{2n}(\Omega S^{2n+1})$ , so

$$u_i u_j = \binom{i+j}{i} u_{i+j}.$$

Using the nontrivial extensions in the Serre spectral sequence, we can choose generators  $e_i \in H^{2ni}(F_{k-1}; Z_{(p)})$  so that

$$(\delta_{k-1})^*(e_i) = \begin{cases} p^{r+d-1}u_i & \text{if } p^{d-1} \leq i < p^d \quad d < k \\ p^{r+k-1}u_i & \text{if } i \geq p^{k-1}. \end{cases}$$

Since  $(\delta_k)^*$  is a monomorphism, it is easy to check that

$$e_1 e_{i-p} \begin{cases} ip^{r-1}e_i & \text{if } i = p^s \quad 0 < s < k \\ ip^r e_i & \text{otherwise.} \end{cases}$$

It now follows from the  $p$ -local cohomology Serre spectral sequence for the fibration

$$S^{2n-1} \longrightarrow W_{k-1} \longrightarrow F_{k-1}$$

that

$$d_{2n}(e_{i-1} \otimes u) = \begin{cases} ip^{r-1}e_i & \text{if } i = p^s \quad 0 < s < k \\ ip^r e_i & \text{otherwise.} \end{cases}$$

From this we can read off the cohomology of  $W_{k-1}$ . □

**Proposition 9.4.** *The homomorphism*

$$H^{2np^k}(W_{k-1}) \longrightarrow H^{2np^k}(T)$$

*is nontrivial of order  $p$ .*

*Proof.* From 9.1 we have a homotopy commutative square

$$\begin{array}{ccc} T & \longrightarrow & \Omega S^{2n+1} \\ \downarrow & & \downarrow \delta_{k-1} \\ W_{k-1} & \longrightarrow & F_{k-1} \end{array}$$

to which we apply cohomology

$$\begin{array}{ccc} H^{2np^k}(T) & \longleftarrow & H^{2np^k}(\Omega S^{2n+1}) \\ \uparrow & & \uparrow \delta_{k-1}^* \\ H^{2np^k}(W_{k-1}) & \longleftarrow & H^{2np^k}(F_{k-1}) \end{array}$$

which we evaluate

$$\begin{array}{ccc} Z/p^{r+k} & \longleftarrow & Z_{(p)} \\ \uparrow & & \uparrow p^{r+k-1} \\ Z/p^{r+k} & \longleftarrow & Z_{(p)} \end{array}$$

where the two horizontal arrows are epimorphisms. The result follows. □



**Proposition 9.5.** *The map  $T \rightarrow R_{k-1}$  extends to a map*

$$T/T^{2np^k-2} \rightarrow R_{k-1}$$

*such that the composition*

$$P^{2np^k}(p^{r+k}) = T^{2np^k}/T^{2np^k-2} \longrightarrow T/T^{2np^k-2} \longrightarrow R_{k-1} \longrightarrow R_k$$

*is null homotopic.*

*Proof.* Since the fibration

$$T \longrightarrow R_{k-1} \longrightarrow G_{k-1}$$

is induced from the fibration over  $G_k$ , we have a homotopy commutative square

$$\begin{array}{ccc} T/\Omega G_{k-1} & \longrightarrow & R_{k-1} \\ \downarrow & & \downarrow \\ T/\Omega G_k & \longrightarrow & R_k. \end{array}$$

Since the inclusion  $T^{2np^k-2} \longrightarrow T$  factors through  $\Omega G_{k-1}$ , this gives a homotopy commutative square

$$\begin{array}{ccc} T/T^{2np^k-2} & \longrightarrow & R_{k-1} \\ \downarrow & & \downarrow \\ T/T^{2np^{k+1}-2} & \longrightarrow & R_k \end{array}$$

The result follows by restriction to  $T^{2np^k}/T^{2np^k-2}$ .  $\square$

**Proposition 9.6.** *Let  $\tilde{\alpha}_k: P^{2np^k}(p^{r+k}) \rightarrow R_{k-1}$  be the composition of the first two maps in 9.5. Then the composition*

$$P^{2np^k}(p^{r+k}) \xrightarrow{\tilde{\alpha}_k} R_{k-1} \longrightarrow W_{k-1}$$

*is nonzero in  $p$  local cohomology.*

*Proof.* This follows from 9.4 using the diagram

$$\begin{array}{ccccc} P^{2np^k}(p^{r+k}) & \longrightarrow & T/T^{2np^k-2} & \longleftarrow & T \\ & \searrow \tilde{\alpha}_k & \downarrow & & \downarrow \\ & & R_{k-1} & \longrightarrow & W_{k-1} \end{array}$$

where the three spaces on the top have isomorphic cohomology in dimension  $2np^k$ .  $\square$

**Proposition 9.7.**  $H_i(F_k; Z_{(p)}) \cong \begin{cases} Z_{(p)} & \text{if } i = 2nm \\ 0 & \text{otherwise.} \end{cases}$

*Proof.* We assume the result for  $F_{k-1}$  by induction. Since  $F_k$  is the total space of a principal fibration over  $D_k = D_{k-1} \cup CP^{2np^k}(p)$  whose restriction to  $D_{k-1}$  is  $F_{k-1}$ , we have by the clutching construction

$$F_k/F_{k-1} = P^{2np^k+1}(p) \rtimes \Omega S^{2n+1};$$

and consequently we have a short exact sequence

$$0 \longrightarrow H_{2nm}(F_{k-1}) \longrightarrow H_{2nm}(F_k) \longrightarrow Z/p \longrightarrow 0$$

for  $m \geq p^k$  while  $H_{2nm}(F_{k-1}) \simeq H_{2nm}(F_k)$  for  $m < p^k$ . We will prove that the extension is nontrivial. It suffices to show that  $H_{2np^k}(F_k) \simeq Z_{(p)}$  since the module action of  $H_*(\Omega S^{2n+1})$  on both  $H_*(F_{k-1})$  and  $H_*(F_k)$  implies the result for all  $m \geq p^k$ . If this failed we would conclude that  $H_{2np^k}(F_k) \cong Z_{(p)} \oplus Z/p$ . This would imply that the homomorphism

$$H^{2np^k}(F_k; Z_{(p)}) \longrightarrow H^{2np^k}(F_{k-1}; Z_{(p)})$$

is onto. We will show that this is impossible. Suppose then that this map is onto and consider the homotopy commutative diagram

$$\begin{array}{ccccccc}
 & & & & & & \Omega S^{2n+1} \\
 & & & & & & \downarrow \delta_k \\
 & & & & & & F_k \\
 & & & & & & \downarrow \sigma_k \\
 P^{2np^k}(p^{r+k}) & \xrightarrow{\tilde{\alpha}_k} & R_{k-1} & \longrightarrow & W_{k-1} & \longrightarrow & F_{k-1} \\
 & & \downarrow & & \downarrow & & \downarrow \sigma_{k-1} \\
 & & G_{k-1} & \longrightarrow & D_{k-1} & \xlongequal{\quad} & D_{k-1} \\
 & & & & & & \downarrow \\
 & & & & & & D_k
 \end{array}$$

The map  $L$  exists since the composition into  $D_k$  factors through the composition

$$R_{k-1} \longrightarrow R_k \longrightarrow G_k \longrightarrow D_k;$$

thus this composition is null homotopic by 9.5. Now if

$$H^{2np^k}(F_k) \longrightarrow H^{2np^k}(F_{k-1})$$

is onto then the entire composition

$$H^{2np^k}(F_k) \longrightarrow H^{2np^k}(P^{2np^k}(p^{r+k}))$$

is nonzero by 9.3 and 9.6. But  $\delta_k$  factors:

$$\Omega S^{2n+1} \xrightarrow{\delta_{k-1}} F_{k-1} \longrightarrow F_k$$

and  $(\delta_{k-1})^*: H^{2np^k}(F_{k-1}) \rightarrow H^{2np^k}(\Omega S^{2n+1})$  is divisible by  $p^{r+k-1}$ . Since  $L^*$  is nonzero on the image of  $\delta_k^*$  which is divisible by  $p^{r+k-1}$  and

$$H^{2np^k}(P^{2np^k}(p^{r+k})) \cong \mathbb{Z}/p^{r+k},$$

we conclude that  $L^*$  is onto. This is impossible for then the composition

$$P^{2np^k}(p^{r+k}) \xrightarrow{L} \Omega S^{2n+1} \xrightarrow{H_{p^{k-1}}} \Omega S^{2np^{k-1}+1}$$

would be onto, where  $H_{p^{k-1}}$  is the James Hopf invariant. But there is never a map

$$P^{2mp}(p^{r+k}) \rightarrow \Omega S^{2m+1}$$

which is onto in cohomology when  $r+k > 1$  since the adjoint

$$P^{2np-1}(p^{r+k}) \longrightarrow \Omega^2 S^{2m+1}$$

would also be onto. Such a map would not commute with the Bockstein. Consequently the extension is nontrivial and the cohomology is free.  $\square$

**Corollary 9.8.** *The induced homomorphism*

$$H_{2ni}(F_{k-1}) \longrightarrow H_{2ni}(F_k)$$

*is an isomorphism when  $i < p^k$  and has degree  $p$  if  $i \geq p^k$ .*  $\square$

This completes the first task of this section. Our second task will be to give a sharper understanding of the spaces  $R_{k-1}$  and, in particular,  $W_{k-1}$ . Recall that  $G_{k-1}$  is a retract of  $\Sigma T^{2np^k-2}$ . Consequently we have a sequence of induced fibrations from 9.1

$$\begin{array}{ccccc} T & \xlongequal{\quad} & T & \xlongequal{\quad} & T \\ \downarrow & & \downarrow & & \downarrow \\ R_{k-1} & \longrightarrow & Q_{k-1} & \longrightarrow & R_{k-1} \\ \downarrow & & \downarrow & & \downarrow \\ G_{k-1} & \longrightarrow & \Sigma T^{2np^k-2} & \longrightarrow & G_{k-1} \end{array}$$

from which we see that  $R_{k-1}$  is a retract of  $Q_{k-1}$ . Using the clutching construction, we see that  $Q_{k-1}$  is homotopy equivalent to a pushout

$$\begin{array}{ccc} T & \xrightarrow{\quad} & Q_{k-1} \\ \uparrow a & & \uparrow \\ T^{2np^k-2} \times T & \longrightarrow & CT^{2np^k-2} \times T \end{array}$$

where  $a$  is the restriction of the action map:

$$T^{2np^k-2} \times T \longrightarrow \Omega G_{k-1} \times T \xrightarrow{a} T.$$

Restricting to  $T^{2np^k-2} \times *$ , we see that the composition

$$T^{2np^k-2} \longrightarrow T \longrightarrow Q_{k-1}$$

is null homotopic, so  $Q_{k-1}/T^{2np^k-2} \simeq Q_{k-1} \vee \Sigma T^{2np^k-2}$ . However, from the pushout diagram, we have

$$Q_{k-1}/T \simeq \Sigma T^{2np^k-2} \rtimes T.$$

Restricting to the  $2np^k - 2$  skeleton, we get

$$Q_{k-1}^{2np^k-2} \vee \Sigma T^{2np^k-2} \simeq (\Sigma T^{2np^k-2} \rtimes T)^{2np^k-2}$$

so  $Q_{k-1}^{2np^k-2} \simeq (\Sigma T^{2np^k-2} \wedge T)^{2np^k-2}$ . Now  $\Sigma T \wedge T$  is a wedge of Moore spaces by (B) in section 2 and only has cells in dimensions congruent to  $-1, 0$ , or  $1 \pmod{2n}$ . Consequently  $Q_{k-1}^{2np^k-2}$  is a wedge of Moore spaces, and the largest exponent is the same as the largest exponent in  $\Sigma T^{2np^k-2}$ , which is  $p^{r+k-1}$ . Since  $R_{k-1}$  is a retract of  $Q_{k-1}$ , we have proved

**Proposition 9.9.**  $R_{k-1}^{2np^k-2}$  is a wedge of  $\text{mod } p^s$  Moore spaces  $P^m(p^s)$  for  $r \leq s < r+k$ .

**Remark.** There are no spheres in this wedge as there are no Moore spaces in  $\Sigma T^{2np^k-2} \wedge T$  of dimension  $2np^k - 1$ .

**Proposition 9.10.** The homomorphism in integral homology

$$H_i(R_{k-1}) \longrightarrow H_i(W_{k-1})$$

is onto for all  $i$  and split for  $i < 2np^k - 1$ .

*Proof.* Since  $D_{k-1}$  is the mapping cone of the composition

$$C_{k-1} = \bigvee_{i=1}^{k-1} P^{2np^k-1}(p^{r+i-1}) \xrightarrow{c} E_{k-1} \xrightarrow{\pi_{k-1}} G_{k-1},$$

there is a clutching construction for the fibrations in 9.1

$$\begin{array}{ccc} T & \xlongequal{\quad} & T \\ \downarrow & & \downarrow \\ R_{k-1} & \longrightarrow & W_{k-1} \\ \downarrow & & \downarrow \\ G_{k-1} & \longrightarrow & D_{k-1} \end{array}$$

and we can describe  $W_{k-1}$  by a pushout diagram

$$\begin{array}{ccc} R_{k-1} & \longrightarrow & W_{k-1} \\ \uparrow & & \uparrow \\ C_{k-1} \times T & \xrightarrow{\pi_2} & T. \end{array}$$

This leads to a long exact sequence

$$\begin{aligned} \dots \longrightarrow \tilde{H}_i(C_{k-1} \times T) &\longrightarrow \tilde{H}_i(R_{k-1}) \otimes \tilde{H}_i(T) \\ &\longrightarrow \tilde{H}_i(W_{k-1}) \longrightarrow \tilde{H}_{i-1}(C_{k-1} \times T). \end{aligned}$$

We assert that the homomorphism

$$\tilde{H}_i(W_{k-1}) \longrightarrow \tilde{H}_{i-1}(C_{k-1} \times T)$$

is trivial. By 9.3,  $\tilde{H}_i(W_{k-1})$  is only nontrivial when  $i = 2sn - 1$  for some  $s \geq 2$ . But  $H_{2sn-2}(C_{k-1} \times T) = 0$  since there are no cells in these dimensions. Now since  $\pi_2: C_{k-1} \times T \rightarrow T$  is onto in homology, we conclude that  $H_i(R_{k-1}) \longrightarrow H_i(W_{k-1})$  is onto. To show that this is split when  $i < 2np^k - 1$ , we note that since  $H_i(W_{k-1})$  is cyclic by 9.3, it suffices to show that the exponent of  $H_i(R_{k-1})$  is not larger than the exponent of  $H_i(W_{k-1})$  for  $i < 2np^k - 1$ . By 9.3, we have

$$\exp(H_{i-1}(W_{k-1})) = \begin{cases} r + \nu_p(i) & i \neq p^s \\ r + s - 1 & i = p^s \quad 0 < s < k. \end{cases}$$

Now

$$\exp(H_{i-1}(R_{k-1})) \leq \exp(H_{i-1}(Q_{k-1})) \leq \exp\left(H_{i-1}\left(\Sigma T^{2np^k-2} \wedge T\right)\right)$$

when  $i - 1 \leq 2np^k - 2$ . However

$$ip^r H_{2ni-1}(\Sigma T \wedge T) = 0$$

and

$$p^{r+s-1} H_{2np^s-1}(\Sigma T \wedge T) = 0. \quad \square$$

**Proposition 9.11.**  $W_{k-1}^{2np^k-2}$  is a wedge of Moore spaces.

*Proof.* Since  $H_{i-1}(R_{k-1}^{2np^k-2}) \rightarrow H_{i-1}(W_{k-1}^{2np^k-2})$  is split onto, we can find a Moore space in the decomposition of  $R_{k-1}^{2np^k-2}$  for each  $i$  representing a given generator. This constructs a subcomplex of  $R_{k-1}^{2np^k-2}$  which is homotopy equivalent to  $W_{k-1}^{2np^k-2}$ .  $\square$

**Corollary 9.12.**  $W_{k-1}^{2np^k-2} \simeq \bigvee_{i=2}^{p^k-1} P^{2ni}(p^{r+n_i})$  where

$$n_i = \begin{cases} v_p(i) & \text{if } i \neq p^s, \quad 0 < s < k \\ s - 1 & \text{if } i = p^s \quad 0 < s < k. \end{cases}$$

## 10. INDUCTIVE CONSTRUCTION

In this section we perform the inductive step of constructing a retraction  $\gamma_k: J_k \rightarrow BW_n$  for  $k \geq 1$ . We presume that  $\gamma_{k-1}$  has been constructed such that the composition

$$\Sigma(\Omega G_{k-1} \wedge \Omega G_{k-1}) \xrightarrow{\Gamma_{k-1}} E_{k-1} \xrightarrow{\tau_{k-1}} J_{k-1} \xrightarrow{\gamma_{k-1}} BW_n$$

is null homotopic. This defines the fiber  $R_{k-1}$  of  $\nu_{k-1} = \gamma_{k-1}\tau_{k-1}$  and we construct  $\beta_k$ ,  $a(k)$  and  $c(k)$  in accordance with 4.4, and  $D_k$ ,  $J_k$  and  $F_k$  as in 6.1.

We next construct a modification of 4.4 in this context.

**Proposition 10.1.** *There is a homotopy commutative ladder of cofibration sequences:*

$$\begin{array}{ccccccc}
 P^{2np^k}(p) & \longrightarrow & P^{2np^k}(p^{r+k}) & \xrightarrow{\sigma} & P^{2np^k}(p^{r+k-1}) & \longrightarrow & P^{2np^k+1}(p) \\
 \uparrow & & \beta_k \downarrow & & a(k) \downarrow & & \uparrow \\
 & & E_{k-1} & & E_k & & \\
 & & \tau_{k-1} \downarrow & & \tau_k \downarrow & & \\
 = & & J_{k-1} & \xrightarrow{\iota} & J_k & & = \\
 & & \eta_{k-1} \downarrow & & \eta_k \downarrow & & \\
 & & F_{k-1} & \longrightarrow & F_k & & \\
 & & \sigma_{k-1} \downarrow & & \sigma_k \downarrow & & \\
 P^{2np^k}(p) & \longrightarrow & D_{k-1} & \longrightarrow & D_k & \longrightarrow & P^{2np^k+1}(p)
 \end{array}$$

*Proof.* The upper central square commutes up to homotopy by 4.4 and 6.3 and the lower central squares follow from (6.2). By a cohomology calculation, the righthand square commutes up to homotopy. For the lefthand region, observe that the  $2np^k$  skeleton of the fiber of the inclusion of  $D_{k-1}$  into  $D_k$  is homotopy equivalent to  $P^{2np^k}(p)$ ; a standard argument with cofibration sequences shows that the lefthand vertical map can be taken to be the identity.  $\square$

**Corollary 10.2.** *The compositions*

$$\begin{aligned}
 P^{2np^k}(p^{r+k-1}) &\xrightarrow{a(k)} E_k \xrightarrow{\tau_k} J_k \xrightarrow{\eta_k} F_k \\
 P^{2np^k}(p^{r+k-1}) &\xrightarrow{\beta_k} E_{k-1} \xrightarrow{\tau_{k-1}} J_{k-1} \xrightarrow{\eta_{k-1}} F_{k-1}
 \end{aligned}$$

*induce integral cohomology epimorphisms.*

*Proof.* The first composition is handled by applying integral cohomology to the righthand region of 10.1. For the second composition we consider the upper two parts of the middle region. The map  $P^{2np^k}(p^{r+k}) \rightarrow P^{2np^k}(p^{r+k-1})$  has degree  $p$  in  $H^{2np^k}$  as does the map  $F_{k-1} \rightarrow F_k$  by 9.8. Since  $r+k \geq 2$ , this is enough to imply the result.  $\square$

**Proposition 10.3.**  $W_{k-1}^{2np^k}$  is a wedge of Moore spaces.

*Proof.* By 9.11,  $W_{k-1}^{2np^k-2}$  is a wedge of Moore spaces. By 9.3, it suffices to show that the map

$$P^{2np^k}(p^{r+k}) \xrightarrow{\beta_k} W_{k-1}^{2np^k}$$

is an epimorphism. But by 10.2, the composition

$$P^{2np^k}(p^{r+k}) \xrightarrow{\beta_k} W_{k-1}^{2np^k} \longrightarrow F_{k-1}$$

induces an isomorphism in  $p$ -local cohomology.  $\square$

We now filter  $F_k$  by skeleta. As in section 8, let  $F_k(m)$  be the  $2mn$  skeleton of  $F_k$ , so

$$F_k(m) = F_k(m-1) \cup e^{2mn}.$$

Let  $J_k(m)$  be the pullback of  $J_k$  to  $F_k(m)$ , so we have a map of principal fibrations

$$(10.4) \quad \begin{array}{ccc} \Omega^2 S^{2n+1} & \xlongequal{\quad} & \Omega^2 S^{2n+1} \\ \downarrow & & \downarrow \\ J_k(m-1) & \longrightarrow & J_k(m) \\ \downarrow & & \downarrow \\ F_k(m-1) & \longrightarrow & F_k(m) \end{array}$$

and using the clutching construction we see that

$$J_k(m)/J_k(m-1) \simeq S^{2mn} \rtimes \Omega^2 S^{2n+1}$$

**Proposition 10.5.** *The compositions*

$$\begin{aligned} P^{2np^k+2ni} &\xrightarrow{\nu^i \overline{a(k)}} J_k \xrightarrow{\eta_k} F_k \\ P^{2np^k+2ni} &\xrightarrow{\mu \nu^{i-1} \overline{a(k)}} J_k(p^k + i - 1) \xrightarrow{q} S^{2(p^k+i-1)n} \rtimes \Omega^2 S^{2n+1} \end{aligned}$$

induce cohomology epimorphisms where  $q$  is the quotient map.

*Proof.* The first composition is evaluated by 10.2 when  $i = 0$ . In case  $i > 0$ , we use induction on  $i$ . We apply 3.11(d) to the diagram

$$\begin{array}{ccc}
 J_k & \xrightarrow{\eta_k} & F_k \\
 \downarrow & & \downarrow \\
 D_k & \xlongequal{\quad} & D_k \\
 \downarrow \varphi'_k & & \downarrow \\
 S^{2n+1}\{p^r\} & \longrightarrow & S^{2n+1}
 \end{array}$$

to see that  $\eta_k(\nu^i \overline{a(k)}) = [\nu, \eta_k \nu^{i-1} \overline{a(k)}]_r$ . The result then follows from 3.12. The second composition is evaluated by using 3.12 directly since  $\mu \nu^i \overline{a(k)} = [\mu, \nu^{i-1} \overline{a(k)}]_r$ .  $\square$

**Corollary 10.6.** *The composition*

$$P^{2np^j+2ni} \xrightarrow{\nu^i \overline{a(j)}} W_j \longrightarrow W_{k-1}$$

*induces a cohomology epimorphism when  $0 \leq i < p^{j+1} - p^j$  and  $j < k$ ; and the composition*

$$P^{2np^j+2n(i+1)-1} \xrightarrow{\mu \nu^i \overline{a(j)}} W_j \longrightarrow W_{k-1}$$

*is nonzero in mod  $p$  cohomology in dimension  $2np^j + 2n(i+1) - 1$ .*

*Proof.* By the induction hypothesis,  $\nu^i \overline{a(j)}$  and  $\mu \nu^{i-1} \overline{a(j)}$  are in the kernel of  $\gamma_j$  for  $j < k$ , so they factor through  $W_j$ . We then construct the diagram

$$\begin{array}{ccc}
 P^{2np^j+2ni} \xrightarrow{\nu^i \overline{a(j)}} W_j & \longrightarrow & W_{k-1} \\
 \downarrow & & \downarrow \\
 F_j & \longrightarrow & F_{k-1}
 \end{array}$$

when  $i < p^k - p^{k-1}$ . Since the map  $F_j \rightarrow F_{k-1}$  induces an isomorphism in cohomology in dimensions less than  $2np^{j+1}$ , the first result follows from 10.5. The second result follows directly from the first since there is a map of fibrations

$$\begin{array}{ccc}
 S^{2n-1} & \longrightarrow & \Omega^2 S^{2n+1} \\
 \downarrow & & \downarrow \\
 W_{k-1} & \longrightarrow & J_{k-1} \\
 \downarrow & & \downarrow \\
 F_{k-1} & \xlongequal{\quad} & F_{k-1}
 \end{array}$$

$\square$



At this point we introduce a simplified notation analogous to the notation in case  $k = 0$ . We define mod  $p^r$  homotopy classes

$$\begin{aligned} x_i(k) &: P^{2ni} \rightarrow J_k \\ y_i(k) &: P^{2ni-1} \rightarrow J_k \end{aligned}$$

for  $i \geq 2$  by the formulas

$$(10.7) \quad \begin{aligned} x_i(k) &= \begin{cases} x_i & \text{if } k = 0 \\ \iota x_i(k-1) & \text{if } i < p^k \\ \nu^{i-p^k} \overline{a(k)} & \text{if } i \geq p^k \end{cases} \\ y_i(k) &= \mu x_{i-1}(k). \end{aligned}$$

Consequently, if  $p^j \leq i < p^{j+1} \leq p^k$ ,  $x_i(k) = x_i(j)$ .

**Corollary 10.8.** *The compositions*

$$P^{2ni} \xrightarrow{x_i(k)} J_k \longrightarrow F_k$$

$$P^{2ni-1} \xrightarrow{y_i(k)} J_k(i-1) \longrightarrow J_k(i-1)/J_k(i-2) \simeq S^{2n(i-1)} \rtimes \Omega^2 S^{2n+1}$$

induce integral cohomology epimorphisms for all  $i \geq 2$ .

*Proof.* This follows from 10.6 and 9.3.  $\square$

Under the inductive hypothesis,  $x_i(k-i)$  and  $y_i(k-1)$  factor through  $W_{k-1}$ .

**Proposition 10.9.** *The map*

$$\Xi: \bigvee_{i=2}^{p^{k-1}} P^{2ni-1} \vee P^{2ni} \xrightarrow{y_i(k-1) \vee x_i(k-1)} W_{k-1}^{2np^k-2}$$

induces a monomorphism mod  $p$  cohomology.

*Proof.*  $H^m(W_{k-1}^{2np^k-2}; Z/p)$  is trivial unless  $m = 2ni$  or  $m = 2ni - 1$  for  $2 \leq i < p^k$ , in which case it is  $Z/p$  by 9.12. Each of these classes is nontrivial under either  $x_i(k-1)$  or  $y_i(k-1)$ .  $\square$

We seek to compare these maps to a natural basis for  $W_{k-1}^{2np^k-2}$ . Choose maps  $e_i: P^{2ni}(p^{r+n_i}) \rightarrow W_{k-1}^{2np^k-2}$  for  $2 \leq i < p^k-1$  which define the splitting of 9.12.

$$e: \bigvee_{i=2}^{p^{k-1}} P^{2ni}(p^{r+n_i}) \xrightarrow{\cong} W_{k-1}^{2np^k-2}$$

Now define a map

$$\Lambda: \bigvee_{i=2}^{p^{k-1}} P^{2ni} \vee S^{2ni-1} \rightarrow W_{k-1}^{2np^k-2}$$

by combining the maps  $e_i \rho^{n_i}: P^{2n_i} \rightarrow W_{k-1}$  and  $e_i \iota_{2n_i-1}: S^{2n_i-1} \rightarrow W_{k-1}$ .

**Proposition 10.10.** *Suppose  $\varphi: \Sigma^2 X \rightarrow W_{k-1}^{2np^k-2}$  has order  $p^r$ . Then  $\varphi$  is congruent to a composition:*

$$\Sigma^2 X \xrightarrow{\varphi'} \bigvee_{i=2}^{p^k-1} P^{2n_i} \vee S^{2n_i-1} \xrightarrow{\Lambda} W_{k-1}^{2np^k-2}.$$

*Proof.* This follows directly from 7.3 and 7.6.  $\square$

**Corollary 10.11.** *If  $n > 1$ , there is a diagram*

$$\begin{array}{ccc} \bigvee_{i=2}^{p^k-1} P^{2n_i} \vee P^{2n_i-1} & & \\ \downarrow & \searrow \Xi & \\ & \equiv & W_{k-1}^{2np^k-2} \\ & \nearrow \Lambda & \\ \bigvee_{i=2}^{p^k-1} P^{2n_i} \vee P^{2n_i-1} & & \end{array}$$

*which commutes up to congruence.*

*Proof.* If  $n > 1$ ,  $\bigvee_{i=2}^{p^k-1} P^{2n_i} \vee P^{2n_i-1}$  is a double suspension whose identity map has order  $p^r$ . Thus 10.11 follows from 10.10.  $\square$

In particular, we obtain a congruence formula by applying 10.11 to the inclusion of  $P^{2n_i}$

$$x_i(k-1) \equiv e_i \rho^{n_i} + \sum_{2 \leq j < i} e_j \rho^{n_j} \alpha_j + e_j \iota_{2n_j-1} \beta_j$$

for some maps  $\alpha_j: P^{2n_i} \rightarrow P^{2n_j}$  and  $\beta_j: P^{2n_i} \rightarrow S^{2n_j-1}$ . Actually, the coefficient of  $e_i \rho^{n_i}$  in this formula is a unit by a cohomology calculation. We can safely assume it is the identity by adjusting the basis  $\{e_i\}$ . We intend to use this formula to replace the term  $e_i \rho^{n_i}$  in  $\Lambda$  by  $x_i(k-1)$ . This is a matter of linear substitutions, and we explain this more clearly in a general context. Observe that all the spaces in these formulas are co- $H$  spaces and 7.5 applies.

**Lemma 10.12.** *In an additive category, the formulas*

$$X = \sum_{i=1}^N a_i \varphi_i + b_i \theta_i$$

$$x_i = a_i + \sum_{j=1}^{i-1} a_j \varphi_{ij} + b_j \theta_{ij}$$

imply that there is a formula:

$$X = \sum_{i=1}^N x_i \bar{\varphi}_i + b_i \bar{\theta}_i.$$

*Proof.* Use downward induction beginning with replacing  $a_N$  with  $x_N$ .  $\square$

**Corollary 10.13.** *Suppose  $\varphi: \Sigma^2 X \rightarrow W_{k-1}^{2np^k-2}$  has order  $p^r$  and  $n > 1$ . Then  $\varphi$  is congruent to a sum*

$$\sum_{i=2}^{p^k-1} x_i(k-1) \bar{\varphi}_i + e_i \iota_{2ni-1} \bar{\theta}_i$$

where  $\bar{\varphi}_i: \Sigma^2 X \rightarrow P^{2ni}$  and  $\bar{\theta}_i: \Sigma^2 X \rightarrow S^{2ni-1}$ .

We would also like to replace  $e_i \iota_{2ni-1}$  by  $y_i(k-1)$ , but this is not as simple. We can use 10.13 to construct a congruence:

$$y_i(k-1) \equiv e_i \beta + \sum_{j < i} x_j(k-1) \bar{\varphi}_j + e_j \iota_{2nj-1} \bar{\theta}_j.$$

Here  $e_i \beta$  is the composition

$$P^{2ni-1} \xrightarrow{\beta} P^{2ni} \xrightarrow{e_i} W_{k-1}$$

while  $e_i \iota_{2ni-1}$  is the composition

$$S^{2ni-1} \xrightarrow{\iota_{2ni-1}} P^{2ni} \xrightarrow{e_i} W_{k-1}.$$

Recall the  $A_*(D_{k-1})$  module structure. Thus for any  $\zeta: P^{m+1} \rightarrow D_{k-1}$ ,  $\zeta \in A_m(D_{k-1})$  and the action is given by the relative Whitehead product. Recall also (7.12), that the module action commutes with compositions. Thus, for example,

$$\zeta x_j(k-1) \bar{\varphi}_j \equiv [\zeta, x_j(k-1) \bar{\varphi}_j]_r \equiv [\zeta, x_j(k-1)]_r \Sigma^m \bar{\varphi}_j.$$

In particular, we have (in the notation of 10.13)

$$[\zeta, \varphi]_r \equiv \sum_{i=2}^{p^k-1} [\zeta, x_i(k-1)]_r \bar{\varphi}'_i + [\zeta, e_i \iota_{2ni-1}]_r \bar{\theta}'_i$$

where  $\bar{\varphi}'_i = \Sigma^m \bar{\varphi}_i$  and  $\bar{\theta}'_i = \Sigma^m \bar{\theta}_i$ .

However, by 6.15,  $[\zeta, e_i \iota_{2ni-1}]_r = [\zeta, e_i \iota_{2ni-1} \pi_{2ni-1}]_r = [\zeta, e_i \beta]_r$ . This gives a formula

$$[\zeta, \varphi]_r \equiv \sum_{i=2}^{p^k-2} [\zeta, x_i(k-1)]_r \overline{\varphi}'_i + [\zeta, e_i \beta]_r \overline{\theta}'_i.$$

We now apply 10.12 again to replace  $[\zeta, e_i \beta]_r$  with  $[\zeta, y_i(k-1)]_r$ . Using the module notation, we have proved

**Proposition 10.14.** *Suppose  $\varphi: \Sigma^2 X \rightarrow W_{k-1}^{2np^k-2}$  has order  $p^r$  and  $n > 1$ . Let  $\zeta: P^{m+1} \rightarrow D_{k-1}$ . Then there is a congruence*

$$\zeta \varphi \equiv \sum_{i=2}^{p^k-1} \zeta x_i(k-1) \widetilde{\varphi}_i + \zeta y_i(k-1) \widetilde{\theta}_i. \quad \square$$

We will use 10.14 to compare the obstructions at adjacent levels. Recall (4.4), the map

$$P^{2np^k}(p^{r+k}) \xrightarrow{\beta_k} R_{k-1} \rightarrow W_{k-1};$$

$\beta_k$  induces a cohomology epimorphism by 10.3. We apply 10.14 where  $\varphi$  is one of the two maps:

$$(10.15) \quad \begin{aligned} \Delta_1 &= \beta_k \rho^k - x_{p^k}(k-1): P^{2np^k} \rightarrow W_{k-1}^{2np^k-2} \\ \Delta_2 &= \beta_k \delta_k - y_{p^k}(k-1): P^{2np^k-1} \rightarrow W_{k-1}^{2np^k-2}. \end{aligned}$$

The maps  $\Delta_1$  and  $\Delta_2$  are uniquely defined as maps to  $W_{k-1}$  since each term lies in the kernel of  $\gamma_{k-1}$ . The fact that they factor through  $W_{k-1}^{2np^k-2}$  follows from 10.3 and 10.5 (In the case  $k = 1$  apply 8.4 and 8.5 in place of 10.5). Note that by 10.1  $\iota \beta_k = a(k) \sigma$  where  $\iota: J_{k-1} \rightarrow J_k$ , so

$$\begin{aligned} \iota \beta_k \rho^k &= a(k) \sigma \rho^k = pa(k) \rho^{k-1} = \overline{pa(k)} \\ \iota \beta_k \delta_k &= a(k) \sigma \delta_k = a(k) \delta_{k-1} = \overline{b(k)}. \end{aligned}$$

Thus we have

$$(10.16) \quad \begin{aligned} \iota \Delta_1 &= \overline{pa(k)} - \iota x_{p^k}(k-1) \\ \iota \Delta_2 &= \overline{b(k)} - \iota y_{p^k}(k-1) \end{aligned}$$

**Theorem 10.17.** *Suppose we have a map  $\gamma_k(m): J_k(m) \rightarrow BW_n$  for  $m \geq p^k$  and  $k \geq 1$  which extends  $\gamma_k(m-1)$  for  $m > p^k$  and  $\gamma_{k-1}$  for  $m = p^k$ . Suppose that*

$$x_j(k) \quad y_{j+1}(k)$$

*are in the kernel of  $\gamma_k(m)$  for  $j \leq m$ . Then the following elements are also in the kernel of  $\gamma_k(m)$ :*

$$(a) \quad \begin{array}{lll} \iota x_j(k-1) & \iota y_{j+1}(k-1) & \text{for } j \leq m \\ (b) \quad \nu^j \overline{b(k)} & \mu \nu^j \overline{b(k)} & \text{for } j + p^k \leq m + 1 \end{array}$$

*Proof.* We first prove (a) by induction on  $m$ . In case  $m = p^k$ , we have by (10.16)

$$px_{p^k}(k) = \iota(x_{p^k}(k-1) + \Delta_1).$$

Since  $BW_n$  has  $H$ -space exponent  $p$ ,

$$\gamma_k(p^k)(\iota(x_{p^k}(k-1) + \Delta_1)) = 0$$

so  $\gamma_k(p^k)(\iota x_{p^k}(k-1)) = -\gamma_k(p^k)(\iota \Delta_1) = 0$ . Applying the action of  $\mu$  gives

$$py_{p^k+1}(k) = \iota(y_{p^k+1}(k-1) + \mu \Delta_1).$$

Now apply 10.14 and (10.15) to get

$$\begin{aligned} \mu \Delta_1 &\equiv \sum_{i=2}^{p^h-1} \mu x_i(k-1) \varphi_i + \mu y_i(k-1) \theta_i \\ &\equiv \sum_{i=2}^{p^k-1} y_{i+1}(k-1) \varphi_i \end{aligned}$$

since  $\mu y_i(k-1) \equiv 0$ . However  $\gamma_{k-1}(y_s(k-1)) = 0$  for  $s \leq p^k$ , consequently

$$\gamma_k(\iota y_{p^k+1}(k-1)) = \gamma_{k-1}(y_{p^k+1}(k-1)) = -\gamma_{k-1}(\mu \Delta_1) = 0.$$

For the inductive step, apply 10.14 with  $\zeta = \nu^d$  or  $\mu \nu^d$  where  $d = m - p^k$ . We get

$$\begin{aligned} \nu^d \Delta_1 &\equiv \sum_{i=2}^{p^k-1} x_{i+d}(k-1) \varphi_{i,d} + y_{i+k}(k-1) \theta_{i,d} \\ \mu \nu^d \Delta_1 &\equiv \sum_{i=2}^{p^k-1} y_{i+d+1}(k-1) \varphi_{i,d}. \end{aligned}$$

These expressions involve  $x_s(k-1)$  for  $s < m$  and  $y_s(k-1)$  for  $s \leq m$ , all of which are in the kernel of  $\gamma_k(m-1)$  by induction. Since

$$px_m(k) \equiv \iota(x_m(k-1) + \nu^d \Delta_1)$$

$$py_{m+1}(k) \equiv \iota(y_{m+1}(k-1) + \mu \nu^d \Delta_1)$$

we see that  $\iota x_m(k-1)$  and  $\iota y_{m+1}(k-1)$  are in the kernel of  $\gamma_k(m)$ .

For part (b), we use (10.16) to get

$$\begin{aligned} \overline{\mu b(k)} &\equiv \iota(\mu y_{p^k}(k-1) + \mu \Delta_2) \\ &\equiv \iota(\mu \Delta_2) \\ &\equiv \iota \left( \sum_{i=2}^{p^k-1} y_{i+1}(k-1) \varphi_i \right). \end{aligned}$$

Since  $\gamma_{k-1}(y_i(k-1)) = 0$  for  $i \leq p^k$ , we have

$$(10.18) \quad \gamma_k(\overline{\mu b(k)}) = 0.$$

Similarly, by (10.16) we have

$$\begin{aligned}\mu\nu^d\overline{b(k)} &\equiv \iota(\mu\nu^d\Delta_2) \\ &\equiv \iota\left(\sum_{i=2}^{p^k-1} y_{i+d+1}(k-1)\varphi_i\right)\end{aligned}$$

which involve  $y_s(k-1)$  for  $s \leq m$ . Likewise

$$\begin{aligned}\nu^d\overline{b(k)} &\equiv \iota(y_m(k-1) + \nu^d\Delta_2) \\ &\equiv \iota\left(y_m(k-1) + \sum_{i=2}^{p^k-1} x_{i+d}(k-1)\tilde{\varphi}_{i,d} + y_{i+d}(k-1)\tilde{\theta}_{i,d}\right).\end{aligned}$$

This involves  $x_s(k-1)$  for  $s < m$  and  $y_s(k-1)$  for  $s \leq m$ , so  $\mu\nu^{d+1}\overline{b(k)}$  and  $\nu^{d+1}\overline{b(k)}$  are in the kernel of  $\gamma_k(m)$ .  $\square$

**Theorem 10.19.** *There is a map  $\gamma: J_k \rightarrow BW_n$  which extends  $\gamma_{k-1}$  on  $J_k(p^k-1) = J_{k-1}(p^k-1)$  and such that the compositions*

$$P^{2np^k}(p^{r+k-1}) \xrightarrow{a(k)} E_k \xrightarrow{\tau_k} J_k \xrightarrow{\gamma_k} BW_n$$

$$P^{2ni} \xrightarrow{x_i(k)} J_k \xrightarrow{\gamma_k} BW_n$$

$$P^{2ni-1} \xrightarrow{y_i(k)} J_k \xrightarrow{\gamma_k} BW_n$$

are null homotopic where  $i \geq 2$ .

*Proof.* We construct  $\gamma_k(m): J_k(m) \rightarrow BW_m$  by induction on  $m$  for  $m \geq p^k$ . We apply 8.1 to the diagram of principal fibrations

$$\begin{array}{ccc}\Omega^2 S^{2n+1} & \longrightarrow & \Omega^2 S^{2n+1} \\ \downarrow & & \downarrow \\ J_k(m-1) & \longrightarrow & J_k(m) \\ \downarrow & & \downarrow \eta_k \\ F_k(m-1) & \longrightarrow & F_k(m)\end{array}$$

for  $m \geq p^k$  and  $k \geq 1$ .  $F_k(m) = F_k(m-1) \cup_{\theta} e^{2mn}$ . Let  $\chi = x_m(k): P^{2mn} \rightarrow J_k(m)$ . By 10.5,  $\eta_k\chi$  induces an isomorphism in mod  $p$  cohomology in dimension  $2mn$ . In case  $m = p^k$ , let  $\gamma_0$  be the restriction of  $\gamma_{k-1}$  to  $J_k(p^k-1)$  and when  $m > p^k$  let  $\gamma_0 = \gamma_k(m-1)$ . By induction, the hypothesis of 10.17 is satisfied for  $m-1$ ; i.e.,  $x_j(k)$  and  $y_{j+1}(k)$  are in the kernel of  $\gamma_k(m-1)$  for  $j < m$ .

We first handle the case  $m = p^k$ . Let

$$P = P^{2np^k}(p^{r+k-1}) \vee P^{2n(p^k+1)-1}$$

and  $u = a(k) \vee \overline{\mu a(k)}: P \rightarrow J_k(p^k)$ . Let

$$P_0 = S^{2np^k-1} \vee S^{2n(p^k+1)-2}$$

where the inclusion  $P_0 \rightarrow P$  is given by the inclusion of the bottom cells. Choose any extension  $\gamma': J_k(p^k) \rightarrow BW_n$  of  $\gamma_{k-1}$ . We now show that the composition

$$P_0 \longrightarrow P \xrightarrow{u} J_k(p^k) \xrightarrow{\gamma'} BW_n$$

is null homotopic. Since  $a(k)\beta = a(k)\sigma\beta\rho = \iota\beta_k\beta\rho$ ,  $\gamma'(a(k)\beta) = 0$  since  $\beta_k$  is in the kernel of  $\gamma_{k-1}$ . Thus the composition

$$S^{2np^k-1} \longrightarrow P^{2np^k}(p^{r+k-1}) \xrightarrow{a(k)} J_k(p^k) \xrightarrow{\gamma'} BW_n$$

is divisible by  $p^{r+k-1}$ . Since  $p\pi_*(BW_n) = 0$ , this composition is null homotopic. Similarly,

$$(\overline{\mu a(k)})\beta = -\mu \left[ (\overline{a(k)}\rho^{k-1})\beta \right] = -\rho^{k-1}\overline{\mu b(k)}.$$

This is in the kernel of  $\gamma_{k-1}$  by (10.18). As before, we conclude that the composition

$$S^{2n(p^k+1)-2} \longrightarrow P^{2n(p^k+1)-1} \xrightarrow{\overline{\mu a(k)}} J_k(p^k) \xrightarrow{\gamma'} BW_n$$

is null homotopic. Finally we check that the composition

$$\begin{aligned} P &\xrightarrow{u} J_k(p^k) \longrightarrow J_k(p^k)/J_k(p^k-1) \\ &\simeq S^{2np^k} \rtimes \Omega^2 S^{2n+1} \xrightarrow{\xi} S^{2np^k} \vee S^{2n(p^k+1)-1} \end{aligned}$$

is an integral cohomology epimorphism by 10.5. Thus, after composing with a homotopy equivalence on the wedge of spheres, we see that this is homotopic to the quotient map  $P \rightarrow P/P_0$ .

The case  $m > p^k$  is similar. In this case we set

$$P = P^{2mn} \vee P^{2(m+1)n-1}$$

$$u = x_m(k) \vee y_{m+1}(k)$$

and we calculate

$$\begin{aligned} x_m(k)\beta &= (\nu^{m-p^k}\overline{a(k)})\beta \\ &= m\mu\nu^{m-p^k-1}\overline{a(k)} + p^{k-1}\nu^{m-p^k}\overline{b(k)} \end{aligned}$$

since  $\overline{a(k)}\beta = a(k)\rho^{k-1}\beta = p^{k-1}a(k)\delta_{k-1}$ . Thus

$$x_m(k)\beta = my_m(k) + p^{k-1}\nu^{m-p^k}\overline{b(k)}.$$

This is in the kernel of  $\gamma_k(m-1)$  by 10.17. Similarly,

$$y_{m+1}(k)\beta = p^{k-1}\mu\nu^{m-p^k}\overline{b(k)}$$

which is also in the kernel of  $\gamma_k(m-1)$  by 10.17. As before, we conclude that the composition

$$\begin{aligned} S^{2mn-1} \vee S^{2(m+1)n-2} &\longrightarrow P^{2mn} \vee P^{2(m+1)n-1} \\ &\xrightarrow{x_m(k) \vee y_{m+1}(k)} J_k(m) \xrightarrow{\gamma'} BW_n \end{aligned}$$

is null homotopic and the composition

$$\begin{aligned} P &\longrightarrow J_k(m) \longrightarrow J_k(m)/J_k(m-1) \\ &\simeq S^{2mn} \rtimes \Omega^2 S^{2n+1} \longrightarrow S^{2mn} \vee S^{2(m+1)n-1} \end{aligned}$$

is equivalent to the quotient map and hence the induction is complete.  $\square$

We now apply 10.17 with  $m = \infty$  to conclude that the classes  $\nu^i\overline{b(k)}$  and  $\mu\nu^i\overline{b(k)}$  are in the kernel of  $\gamma_k$  for all  $i \geq 0$ , and that for all  $i \geq 2$ , the classes  $x_i(k-1)$  and  $y_i(k-1)$  are in the kernel of the composition:

$$J_{k-1} \xrightarrow{\iota} J_k \xrightarrow{\gamma_k} BW_n.$$

It follows that we can reapply 10.17 to  $J_{k-1}$  and by continuing, conclude that all the classes  $x_i(j)$ ,  $y_i(j)$ ,  $\nu^i\overline{b(j)}$ ,  $\mu\nu^i\overline{b(j)}$  are in the kernel of  $\gamma_n$  for  $j \leq k$ . We conclude

**Corollary 10.20.** *The composition*

$$\Sigma(\Omega G_{k-1} \wedge \Omega G_k) \xrightarrow{\Gamma_k} E_k \xrightarrow{\tau_k} J_k \xrightarrow{\gamma_k} BW_n$$

*is null homotopic.*

*Proof.* By 7.15, all congruence classes of the obstructions listed in 6.13 are annihilated by  $\gamma_k$ . It follows that the compositions

$$\Sigma P_k \wedge \cdots \wedge P_k \xrightarrow{\widetilde{ad}_k^j(a)} J_k \xrightarrow{\gamma_k} BW_n$$

are null homotopic for each  $j \geq 2$ . The result then follows from 5.11.  $\square$

This completes the induction, and we have proved the Main Theorem.  $\square$

**Theorem 10.21.** *If  $p > 3$ ,  $T$  has  $H$ -space exponent  $p^r$ .*

*Proof.* We will show that the  $p^{r\text{th}}$  power map and the constant map are homotopic. Since  $T$  is Abelian, they are both  $H$ -maps. We appeal to the following result:  $\square$

**Proposition 10.22.** [Gra11, 5.4] *Suppose  $f_1, f_2: T \rightarrow Z$  are two  $H$ -maps extending  $f: P^{2n} \rightarrow Z$ . Suppose that  $p^{r+s-1}\pi_{2np^s}(Z) = 0$  and the torsion subgroup of  $\pi_{2np^s-1}(Z)$  has exponent  $p^{r+s-1}$  for each  $s \geq 1$ . Then  $f_1 \sim f_2$ .*



We apply this with  $f_1 = p^r$  and  $f_2 = *$ . It suffices to prove the exponent conditions on the homotopy groups. The long exact sequence

$$\pi_i(W_n) \rightarrow \pi_i(T) \rightarrow \pi_{i+1}(S^{2n+1}\{p^r\}) \rightarrow$$

implies that  $p^{r+1}\pi_i(T) = 0$ . Consequently we need only verify that  $p^r\pi_{2np}(T) = 0$  and  $p^r\pi_{2np-1}(T) = 0$ . However  $\pi_{2np-1}(W_n) = 0$ , and, if  $p > 3$ ,  $\pi_{2np}(W_n) = 0$ . Consequently  $p^r\pi_{2np}(T) = 0$  and  $p^r\pi_{2np-1}(T) = 0$  and 10.21 follows from 10.22.

Now let  $R = \bigcup R_k$ ,  $W = \bigcup W_k$  and  $C = \bigcup C_k$ .

**Proposition 10.23.**  $R \simeq C \rtimes T \vee W$ .

*Proof.* By 9.10, the homomorphism

$$H_i(R) \longrightarrow H_i(W)$$

is a split epimorphism. Since  $W$  is a wedge of Moore spaces by 9.12, and  $R$  is a wedge of Moore spaces by 9.9, it follows that the map  $R \rightarrow W$  has a left homotopy inverse. Consider the pushout diagram

$$\begin{array}{ccc} R & \longrightarrow & W \\ \uparrow & & \uparrow \\ C \times T & \xrightarrow{\pi_2} & T \end{array}$$

obtained from the proof of 9.10 by taking limits. Using the fact that the map  $T \rightarrow W$  is null homotopic we get a cofibration sequence

$$C \rtimes T \longrightarrow R \longrightarrow W$$

which splits. □

**Corollary 10.24.** *The map  $\Omega G \rightarrow \Omega D$  has a right homotopy inverse and the fiber of the map  $G \rightarrow D$  is  $C \rtimes \Omega D$ .*

*Proof.* Since  $R \rightarrow W$  has a right homotopy inverse, the map

$$\Omega G \simeq \Omega R \times T \rightarrow \Omega W \times T \simeq \Omega D$$

does as well. Let  $L$  be the fiber of the map  $G \rightarrow D$ . Then there is a diagram of fibrations:

$$\begin{array}{ccccc} \Omega D & \xlongequal{\quad} & \Omega D & \xlongequal{\quad} & \Omega D \\ \downarrow & & \downarrow & & \downarrow \\ C \times \Omega D & \longrightarrow & L & \longrightarrow & PD \\ \downarrow & & \downarrow & & \downarrow \\ C & \longrightarrow & G & \longrightarrow & D. \end{array}$$

Since  $D$  is a mapping cone,  $PD$  is the pushout

$$\begin{array}{ccc} L & \longrightarrow & PD \\ \uparrow & & \uparrow \\ C \times \Omega D & \xrightarrow{\pi_2} & \Omega D. \end{array}$$

Since  $\Omega G \rightarrow \Omega D$  has a right homotopy inverse, the map  $\Omega D \rightarrow L$  is null homotopic. Consequently we obtain a map  $C \rtimes \Omega D \rightarrow L$  which is a homotopy equivalence.  $\square$

**Proposition 10.25.** *The composition*

$$C \rtimes T \longrightarrow R \longrightarrow G$$

*is homotopic to the Whitehead product map*

$$C \rtimes T \rightarrow C \rtimes \Omega G \rightarrow C \vee G \xrightarrow{c \vee 1} G.$$

*Proof.* The map  $C \rtimes T \rightarrow R$  is given by

$$C \rtimes T \simeq C \times T \cup CT \rightarrow R$$

where the restriction to  $C \times T$  is the trivialization of the pullback to  $C$

$$\begin{array}{ccccc} T & \longrightarrow & T & \xlongequal{\quad} & T \\ \downarrow & & \downarrow & & \downarrow \\ C \times T & \longrightarrow & R & \longrightarrow & W \\ \downarrow & & \downarrow & & \downarrow \\ C & \longrightarrow & G & \longrightarrow & D \end{array}$$

and the map on  $CT$  is the null homotopy of the inclusion of  $T$  into  $R$ . Consequently the composition

$$C \times T \cup CT \longrightarrow R \longrightarrow G$$

is given on  $C \times T$  by projection onto  $C$  followed by the inclusion  $c: C \rightarrow G$ , and on  $CT$  by the composition

$$CT \longrightarrow \Sigma T \xrightarrow{\tilde{g}} G$$

where  $\tilde{g}$  is adjoint to the map lifting  $T$  to  $\Omega G$ . However, this map factors through the composition

$$\begin{array}{ccccccc} C \rtimes \Omega \Sigma T \simeq C \times \Omega \Sigma T \cup_{\Omega \Sigma T} P \Sigma T & \longrightarrow & C \vee \Sigma T & \longrightarrow & G \\ \downarrow & & \downarrow & \nearrow c \vee 1 & \\ C \rtimes \Omega G \simeq C \times \Omega G \cup_{\Omega G} P G & \longrightarrow & C \vee G & & \end{array} \quad \square$$

## APPENDIX A.

In this appendix we will discuss the special cases  $n = 1$  and  $p = 3$ .

**Theorem A.1.** *For all  $p > 2$  and all  $r \geq 1$   $T_1(p^r)$  is homotopy equivalent to a double loop space, hence it is Abelian.*

*Proof.* Let  $\mu \in H^4(BS^3)$  be a generator and let  $\kappa = p^r \mu$  for any  $p$ . Define  $X$  by the fibration:

$$K(Z, 3) \longrightarrow X \longrightarrow BS^3 \xrightarrow{\kappa} K(Z, 4).$$

Then  $H_3(X) = Z/p^r$ . Since  $X$  is 2 connected, there is a map

$$\xi: P^4(p^r) \rightarrow X$$

inducing an isomorphism in  $H_3$ . By [GT10, 4.3(i)],  $\Sigma G_k$  is a wedge of Moore spaces for all  $k$ , so we can construct a map

$$\eta: \Sigma^2 T_1(p^r) \rightarrow \Sigma G \rightarrow P^4(p^r)$$

which induces an isomorphism in  $\pi_3$ . We now define a map

$$T_1(p^r) \xrightarrow{\tilde{\eta}} \Omega^2 P^4(p^r) \xrightarrow{\Omega^2 \xi} \Omega^2 X$$

inducing an isomorphism in  $\pi_1$ .

Both spaces have mod  $p$  cohomology generated by a class  $n$  of dimension 1 and  $v_i$  of dimension  $2p^i$  for  $i \geq 0$  with  $\beta^{(r+i)}(uv_0^{p-1}v_1^{p-1}\dots v_{i-1}^{p-1}) = v_i$  by a simple spectral sequence argument (See [GT10, 4.1]). Since the composition  $(\Omega^2 \xi)\tilde{\eta}$  induces an isomorphism in dimensions 1 and 2, it follows that  $((\Omega^2 \xi)\tilde{\eta})_*$  is an isomorphism in each dimension. Since both spaces are  $p$ -complete,  $(\Omega^2 \xi)\tilde{\eta}$  is a homotopy equivalence.  $\square$

**Theorem A.2.** *If  $T_{2n-1}(3^r)$  is homotopy associative,  $n = 3^k$  with  $k \geq 0$ . Furthermore if  $n > 1$ , then  $r = 1$ .*

*Proof.* For any homotopy associative space  $T$ , there is a map:

$$T * T * T \rightarrow ST \cup_{H(\mu)} C(T * T)$$

building the third stage of the classifying space construction ([Sug57], [Sta63]). The mapping cone  $X$  of this map has the cohomology of the bar construction on the homology of  $T$  through dimension  $8n - 1$ . In particular the mod  $p$  cohomology of the  $6n$  skeleton of  $X$  has as a basis, classes  $u, v, u^2, uv, u^3$  where  $|u| = 2n$  and  $|v| = 2n + 1$ . The  $6n$  skeleton of the subspace  $ST \cup_{H(\mu)} CT * T$  has cohomology generated by  $u, v, u^2, uv$ . Now since the map  $R \rightarrow G$  is a retract of the map  $H(\mu): T * T \rightarrow \Sigma T$ , the  $4n + 1$  skeleton of  $ST \cup_{H(\mu)} CT * T$  contains the  $4n + 1$  skeleton of  $G \cup CR$  as a retract (See the proof of 9.8). But  $[G \cup CR]^{4n} = P^{2n+1} \cup_{x_2} CP^{4n}$ ; consequently

$$X^{6n} \simeq P^{2n+1} \cup_{x_2} CP^{4n} \cup e^{6n}.$$

Note that  $\mathcal{P}^n u = u^3$  generates  $H^6(X; \mathbb{Z}/3)$ . Since  $\Sigma x_2$  is inessential we can pinch the middle cells to a point after one suspension, and obtain a space with cell structure

$$S^{2n+1} \cup_{p^r} e^{2n+2} \cup e^{6n+1}$$

with  $\mathcal{P}^n \neq 0$ . However,  $\mathcal{P}^n$  is decomposable unless  $n = p^k = 3^k$ . Furthermore, the decomposition of  $\mathcal{P}^{p^k}$  by secondary operations ([Liu62b]) implies that if  $n > 1$ , we must have  $r = 1$ .  $\square$

Note that such a space for  $n > 1$  would imply that the “mod 3 Arf invariant class” survives the Adams spectral sequence. This does happen when  $n = p$  but not when  $n = p^2$ .

**Theorem A.3.**  *$T_{2p-1}(p)$  is homotopy equivalent to a double loop space.*

*Proof.* In [Tod56], a  $p$ -local fibration sequence

$$J_{p-1}(S^2) \longrightarrow \Omega S^3 \xrightarrow{H} \Omega S^{2p+1}$$

where  $H$  is the James Hopf invariant and  $J_{p-1}(S^2)$  is the subset of the James construction of words of length less than  $p$ . There is a  $p$ -local equivalence  $J_{p-1}(S^2) \simeq CP^{p-1}$  and if we take the 2-connected cover, we get the fibration sequence:

$$S^{2p-1} \longrightarrow \Omega S^3 \langle 2 \rangle \longrightarrow \Omega S^{2p-1}.$$

Here the connecting homomorphism has degree  $p$ . Since

$$\Omega S^3 \langle 2 \rangle = \Omega^2(BS^3 \langle 4 \rangle),$$

we have a double loop space. As in A.1, we construct  $P^{2p+2}(p) \rightarrow BS^3 \langle 4 \rangle$  inducing an isomorphism in  $\pi_{2p+1}$  and then compose

$$T_{2p-1}(p) \xrightarrow{\tilde{\eta}} \Omega^2 P^{2p+2}(p) \xrightarrow{\Omega^2 \tilde{\xi}} \Omega S^3 \langle 2 \rangle.$$

The rest of the proof is identical with A.1 and this establishes the homotopy equivalence  $T_{2p-1}(p) \simeq \Omega^2(BS^3 \langle 4 \rangle)$ .  $\square$

## APPENDIX B.

The purpose of this appendix is to indicate how the ideas and techniques in this work could be applied in other situations. Suppose we are given an arbitrary space which we know admits an  $H$ -space structure, and we wish to inquire as to whether there is an Abelian  $H$ -space structure. We will assume that all the spaces are  $p$ -complete. The following theorem appears in [Gra06].

**Theorem B.1.** *In the category of  $p$ -complete spaces there is a 1–1 correspondence between atomic spaces  $T$  which admit an  $H$ -space structure and atomic simply connected spaces  $G$  which admit a co- $H$  space structure. Furthermore*

(a) *There are maps  $f: G \rightarrow \Sigma T$ ,  $g: T \rightarrow \Omega G$ , and  $h: \Omega G \rightarrow T$  such that the compositions*

$$G \xrightarrow{f} \Sigma T \xrightarrow{\tilde{g}} G$$

$$T \xrightarrow{g} \Omega G \xrightarrow{h} T$$

*are homotopic to the identity*

(b) *there is a fibration*

$$\Omega G \xrightarrow{h} T \longrightarrow R \xrightarrow{\pi} G$$

*induced by  $f$  from the Hopf fibration of Dold–Lashof ([DL59]), where  $R$  is a retract of  $\Sigma(T \wedge T)$  and hence is a co- $H$  space.*

Note that the homotopy type of  $G$  is determined by that of  $T$ . It is not clear that the same thing can be said about  $R$ . However, if we have two such fibrations

$$\Omega G \xrightarrow{h_1} T \xrightarrow{i_1} R_1 \xrightarrow{\pi_1} G$$

$$\Omega G \xrightarrow{h_2} T \xrightarrow{i_2} R_2 \xrightarrow{\pi_2} G$$

we have  $T \times \Omega R_1 \simeq \Omega G \simeq T \times \Omega R_2$ . Since the homology of all three loop spaces is a tensor algebra, we can see that the Poincaré polynomials of  $R_1$  and  $R_2$  are equal. We can also produce maps going both ways using the co- $H$  space structure maps on  $R_1$  and  $R_2$ .

It is clear, however, that the map  $\pi$  depends on the choice of  $H$ -space structure. We see this from the following example. Let  $B$  be the localization of  $B\Sigma_p$  at the prime  $p$ . Then

$$H^*(B; \mathbb{Z}/p) \cong \mathbb{Z}/p[v, 2p-2] \otimes \Lambda(u, 2p-3)$$

and there is a fibration

$$S^{2p-3}\{p\} \longrightarrow S^{2p-3} \xrightarrow{p} S^{2p-3} \longrightarrow B.$$

In this fibration  $S^{2p-3}\{p\} \simeq \Omega B$ . Applying B.1 to this fibration we get a fibration

$$S^{2p-3}\{p\} \longrightarrow R \xrightarrow{\pi} P^{2p-2}(p) \longrightarrow B.$$

However, the presence of nontrivial Steenrod operations in  $H^*(B; \mathbb{Z}/p)$  implies that  $\pi$  is stably essential. This is certainly different from the fibration in 7.6 [CMN79c, 11.1] in which the map  $\pi$  is a wedge of iterated Whitehead products.

The problem of constructing a map  $\Omega G * \Omega G \rightarrow R$  covering the universal Whitehead product as in 2.3 is that we need first to choose an appropriate

map  $\pi: R \rightarrow G$ . This was circumvented in the Main Theorem by constructing first the fibration:

$$\Omega S^{2n+1}\{p^r\} \longrightarrow E \longrightarrow G \xrightarrow{\varphi} S^{2n+1}\{p^r\}.$$

Then the map  $\pi: R \rightarrow G$  was determined by the choice of a map  $E \rightarrow BW_n$ .

This approach can be applied in other situations. Suppose we begin with an arbitrary  $p$ -complete atomic space  $T$  which admits an  $H$ -space structure, and for some such structure we are given an  $H$ -map  $e: T \rightarrow \Omega X$ . Choose appropriate maps  $f, g, h$  as in B.1 and consider the diagram of fibration

$$\begin{array}{ccccc} T & \xlongequal{\quad} & T & \xrightarrow{e} & \Omega X \\ \downarrow & & \downarrow & & \downarrow \\ R & \longrightarrow & T * T & \xrightarrow{e * e} & \Omega X * \Omega X \\ \pi \downarrow & & \downarrow H(\mu) & & \downarrow H(\mu') \\ G & \xrightarrow{f} & \Sigma T & \xrightarrow{\Sigma e} & \Sigma \Omega X \xrightarrow{\epsilon} X \end{array}$$

where  $\mu'$  is the loop space multiplication on  $\Omega X$ . Let  $\varphi: G \rightarrow X$  be the composition  $\epsilon(\Sigma e)f$ . Since  $\epsilon H(\mu')$  is null homotopic,  $\varphi\pi$  is null homotopic and we construct a map of fibration:

$$\begin{array}{ccc} T & \xrightarrow{e} & \Omega X \\ \downarrow & & \downarrow \\ R & \longrightarrow & E \\ \downarrow & & \downarrow \\ G & \xlongequal{\quad} & G \\ & & \downarrow \varphi \\ & & X. \end{array}$$

In fact, the existence of such a fibration implies that  $e$  is an  $H$ -map (See [GT10, 4.6] for details). If we choose  $X$  to be an  $H$ -space, we would automatically have a lifting of the universal Whitehead product

$$\Gamma: \Omega G * \Omega G \rightarrow E.$$

We could construct such a map  $e$  by taking the adjoint of the composition

$$\Sigma T \xrightarrow{\tilde{g}} G \longrightarrow \Omega \Sigma G \xrightarrow{\xi} X$$

where  $\xi$  is a retraction onto an atomic factor of least connectivity. (In the case of the Anick spaces,  $\Sigma G$  is a wedge of Moore spaces ([GT10, 9.5]) and the atomic factor of least connectivity is  $S^{2n+1}\{p^r\}$ .)

In the general case we need a replacement for  $BW_n$ . Let  $W$  be the fiber of  $e$ . The task then is to construct the map  $\nu$  for some choice of  $h$  in B.1 so

that in the diagram

$$\begin{array}{ccccc}
 \Omega G \wedge \Omega G & \xrightarrow{\tilde{\Gamma}} & \Omega E & \xrightarrow{-\nu-} & W \\
 & \searrow \widetilde{\nabla\omega} & \downarrow & & \downarrow \\
 & & \Omega G & \xrightarrow{-h-} & T \\
 & & \downarrow \Omega\varphi & & \downarrow e \\
 & & \Omega X & \xlongequal{\quad} & \Omega X.
 \end{array}$$

$\nu\tilde{\Gamma}$  is null homotopic. If that is done,  $h\widetilde{\nabla\omega}$  is null homotopic and  $\widetilde{\nabla\omega}$  will lift to the fiber of  $h$ . Since  $h$  is to be constructed to have a right homotopy inverse  $g$ , the fiber of  $h$  is  $\Omega R$  and the lifting of  $\widetilde{\nabla\omega}$  to  $\Omega R$  gives the requisite lifting of  $\nabla\omega$  to  $R$ .

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